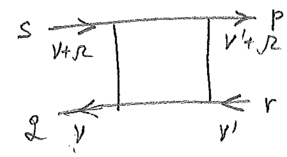
TWO - PARTICLE VERTEX NOTES

For other details see Hyowon's thesis.

Local vertex sampled in ctqmc

$$\chi_{sq;pr} \equiv \langle \psi_p^+(\tau_1)\,\psi_q^+(\tau_2)\,\psi_r(\tau_3)\,\psi_s(\tau_4)\rangle$$



In ctqmc we evaluate vertex in the following way:

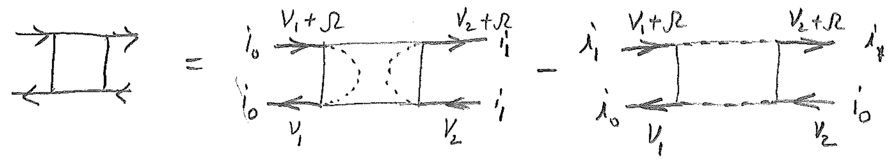
$$\chi H[i_0, i_1;\, i\Omega\; i\nu_1\; i\nu_2] = M[i_0](i\nu_1, i\nu_1+i\Omega)\; M[i_1](i\nu_2+i\Omega, i\nu_2)$$

$$\chi F[i_0, i_1;\, i\Omega\; i\nu_1\; i\nu_2] = M[i_0](i\nu_1\; i\nu_2)\; M[i_1](i\nu_2+i\Omega, i\nu_1+i\Omega)$$

where $M[i_0](i\nu, i\nu) = G_{i_0}(i\nu)$ is the one particle Green's function.

Diagrammatically this is:

Vertex:



The vertex can be derived by the formula:

$$\chi_{sq;pr} = \frac{1}{Z}\frac{\partial^2 Z}{\partial\Delta_{ps}\,\partial\Delta_{qr}} = \frac{1}{Z}\frac{\partial^2}{\partial\Delta_{ps}\,\partial\Delta_{qr}}\int \mathcal{D}[\psi^+\psi]\, e^{-S_{atom} - \sum_{\alpha\beta}\int \psi_\alpha^+ \Delta_{\alpha\beta}\psi_\beta}$$

$$= \frac{1}{Z}\int \mathcal{D}[\psi^+\psi]\, e^{-S_{atom}}\, \psi_p^+\psi_s\,\psi_q^+\psi_r$$

$$= \frac{1}{Z}\frac{\partial^2}{\partial\Delta_{ps}\,\partial\Delta_{qr}}\int \mathcal{D}[\psi^+\psi]\, e^{-S_{atom}}\,(\psi_{\alpha_1}^+\psi_{\alpha_2}\,\psi_{\alpha_3}\,\psi^+_{\dots})\,\mathrm{Det}(\Delta)$$

$$\chi_{sq;pr} = \frac{1}{Z}\,\mathrm{Tr}(\psi_1^+\psi_2\,\psi_3\,\psi^+_{\dots})\,\frac{\partial^2}{\partial\Delta_{ps}\,\partial\Delta_{qr}}\,\mathrm{Det}(\Delta)$$

In practice, we print out $\chi - \chi^0$, where χ^0 is bubble. In particular

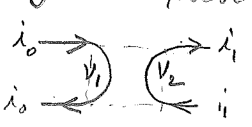
$$\chi^0 \equiv B_H + B_F$$

$$B_H[i_0, i_1, i\Omega\; i\nu_1\; i\nu_2] = \delta(\Omega=0)\, M[i_0](i\nu_1\, i\nu_1)\, M[i_1](i\nu_2\, i\nu_2) \equiv \delta(\Omega=0)\, G_{i_0}(i\nu_1)\, G_{i_1}(i\nu_2)$$

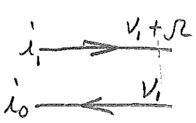
$$B_F[i_0\; i_1,\; i\Omega\; i\nu_1\; i\nu_2] = \delta(\nu_1=\nu_2)\, M[i_0](i\nu_1)\, M[i_1](i\nu_1+i\Omega) \equiv \delta(\nu_1=\nu_2)\, G_{i_0}(i\nu_1)\, G(i\nu_1+i\Omega)$$

Diagrammatically we have:

$B_H =$



$B_F =$



Magnetic and charge χ

$\chi = \chi_0 + \chi_0 \Gamma \chi$: Dyson equation $\rightarrow$ Γ is irreducible vertex
χ_0 is the bubble

$$\chi_{ss;s's'} = \chi^0_{ss}\delta_{ss'} + \chi^0_{ss}\,\Gamma_{ss;s''s''}\,\chi_{s''s'',s's'}$$

In paramagnetic state we have $\chi_{\uparrow\uparrow;\uparrow\uparrow} = \chi_{\downarrow\downarrow;\downarrow\downarrow}$ and $\chi_{\uparrow\uparrow;\downarrow\downarrow} = \chi_{\downarrow\downarrow;\uparrow\uparrow}$ hence

$$\chi_{\uparrow\uparrow;\uparrow\uparrow} = \chi^0_{\uparrow\uparrow} + \chi^0_{\uparrow\uparrow}\left(\Gamma_{\uparrow\uparrow;\uparrow\uparrow}\,\chi_{\uparrow\uparrow;\uparrow\uparrow} + \Gamma_{\uparrow\uparrow;\downarrow\downarrow}\,\chi_{\downarrow\downarrow;\uparrow\uparrow}\right)$$

$$\chi_{\uparrow\uparrow;\downarrow\downarrow} = \emptyset + \chi^0_{\uparrow\uparrow}\left(\Gamma_{\uparrow\uparrow;\uparrow\uparrow}\,\chi_{\uparrow\uparrow;\downarrow\downarrow} + \Gamma_{\uparrow\uparrow;\downarrow\downarrow}\,\chi_{\downarrow\downarrow;\downarrow\downarrow}\right)$$

we have:

$$\chi_{\uparrow\uparrow;\uparrow\uparrow} \pm \chi_{\uparrow\uparrow;\downarrow\downarrow} = \chi^0_{\uparrow\uparrow} + \chi^0_{\uparrow\uparrow}\left\{\Gamma_{\uparrow\uparrow;\uparrow\uparrow}\left(\chi_{\uparrow\uparrow;\uparrow\uparrow} \pm \chi_{\uparrow\uparrow\downarrow\downarrow}\right) + \Gamma_{\uparrow\uparrow\downarrow\downarrow}\left(\chi_{\downarrow\downarrow\uparrow\uparrow} \pm \chi_{\downarrow\downarrow\downarrow\downarrow}\right)\right\}$$

$\chi^d \equiv \chi_{\uparrow\uparrow\uparrow\uparrow} + \chi_{\uparrow\uparrow\downarrow\downarrow}$ and $\Gamma^m = \chi_{\uparrow\uparrow\uparrow\uparrow} - \chi_{\uparrow\uparrow\downarrow\downarrow}$

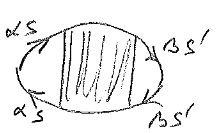
$\Gamma^d = \Gamma_{\uparrow\uparrow\uparrow\uparrow} + \Gamma_{\uparrow\uparrow\downarrow\downarrow}$ and $\Gamma^m = \Gamma_{\uparrow\uparrow\uparrow\uparrow} - \Gamma_{\uparrow\uparrow\downarrow\downarrow}$

finally:

$$\boxed{\begin{aligned}\chi^m &= \chi^0 + \chi^0\,\Gamma^m\,\chi^m\\ \chi^d &= \chi^0 + \chi^0\,\Gamma^d\,\chi^d\end{aligned}}$$

In paramagnetic state we compute $\langle M^z(\tau) M^z(0)\rangle$. If spin-orbit is small, we have $\langle S^z(\tau) S^z(0)\rangle$

$$\chi^{zz} = \sum_{\alpha\beta}\left(\chi_{\alpha\uparrow\alpha\uparrow;\beta\uparrow\beta\uparrow} + \chi_{\alpha\downarrow\alpha\downarrow;\beta\downarrow\beta\downarrow} - \chi_{\alpha\uparrow\alpha\uparrow;\beta\downarrow\beta\downarrow} - \chi_{\alpha\downarrow\alpha\downarrow;\beta\uparrow\beta\uparrow}\right)(S^z)^2$$

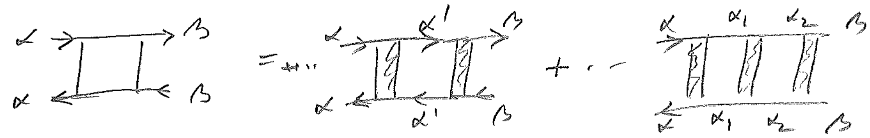


$$= \sum_{\alpha\beta} 2\chi^m_{\alpha\alpha\,\beta\beta}\cdot\frac{1}{4}$$

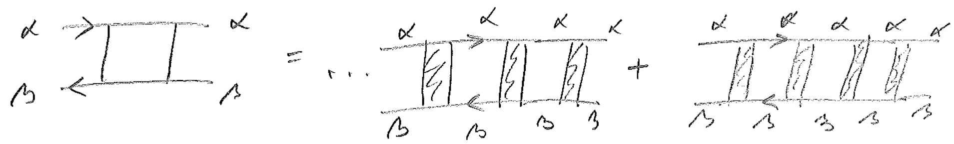
From [illegible] we have

$\chi^{\text{impurity}}_{\alpha\alpha\beta\beta}$ and $\chi^{\text{impurity}}_{\alpha\beta\alpha\beta}$

If we consider only the z-component of Hunds coupling, we have α as a "good local" quantum number, hence we have



and also



We thus have $\chi_{\alpha_1\alpha_2;\beta_1\beta_2} = \chi^{(1)}_{\alpha\alpha;\beta\beta}\,\delta_{\alpha_1=\alpha_2}\,\delta_{\beta_1=\beta_2} + \chi^{(2)}_{\alpha\beta;\alpha\beta}\,\delta_{\alpha_1=\beta_1=\alpha}\,\delta_{\alpha_2=\beta_2=\beta}$

here $\alpha \neq \beta$ to avoid double-counting

The impurity ones $\chi^{(1)}$ and $\chi^{(2)}$ satisfy:

$$\chi^{(1)}_{\alpha\alpha;\beta\beta} = \chi^0_{\alpha\alpha}\,\delta_{\alpha\beta} - \chi^0_{\alpha\alpha}\,\Gamma_{\alpha\alpha\alpha'\alpha'}\,\chi^{(1)}_{\alpha'\alpha'\beta\beta}$$

hence we invert matrix in frequency $\nu\nu'$ and orbitals $\alpha\beta$ to obtain

$$(\Gamma^{(1)})_{\alpha\alpha\nu;\beta\beta\nu'} = \left(\chi^{-1} - \chi_0^{-1}\right)^{-1}_{\alpha\alpha\nu;\beta\beta\nu'}$$

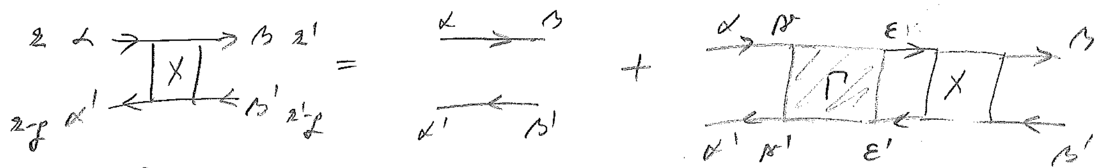
For $\chi^{(2)}_{\alpha\beta\alpha\beta}$ when $\alpha \neq \beta$ we simply have

$$\chi^{(2)}_{\alpha\beta\alpha\beta} = \chi^0_{\alpha\beta\alpha\beta} - \chi^0_{\alpha\beta\alpha\beta}\,\Gamma_{\alpha\beta\alpha\beta}\,\chi^{(2)}_{\alpha\beta\alpha\beta}$$

hence we have matrix multiplication only in frequency ν and ν' but not in orbital index. We thus have;

$$\Gamma^{(2)}_{\alpha\beta\alpha\beta} = \left(\chi^{-1}_{\alpha\beta\alpha\beta} - \chi^{0\,-1}_{\alpha\beta\alpha\beta}\right)^{-1}$$ with matrices in frequency only

To compute q-dependent susceptibility, we need to work with all 4 orbital index χ.



$$\chi^{q}_{\alpha\alpha'\beta\beta';\,k\,k'} = \chi^{0\,q}_{\alpha\alpha'\beta\beta';\,k\,k}\,\delta_{k-k'} - \chi^{0\,q}_{\alpha\alpha'\mu\mu';\,k\,k}\,\delta_{k-k'}\,\Gamma_{\mu\mu'\varepsilon\varepsilon'}\,\chi^{q}_{\varepsilon\varepsilon'\beta\beta';\,k'}$$

Since Γ is momentum independent, we can sum over internal momenta and define

$$\chi^{0\,q}_{\alpha\alpha'\beta\beta'} = \sum_k \chi^{0\,q}_{\alpha\alpha'\beta\beta';\,k\,k} = \sum_k G^{k}_{\beta\alpha}(i\nu)\,G^{k-q}_{\alpha'\beta'}(i\nu - i\Omega)$$

hence we have only momentum q as parameter. We keep full frequency dependence and orbital dependence.

We define combined index $(\alpha\alpha'\nu)$ and write

$$\chi^{q,\Omega}_{(\alpha\alpha'\nu)(\beta\beta'\nu')} = \left[\chi^{0\,-1}_{q\Omega} - \Gamma_\Omega\right]^{-1}_{(\alpha\alpha'\nu)(\beta\beta'\nu')}$$ here q and Ω are parameters

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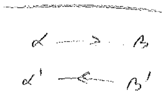
On real axis we need $\chi^0_q(\Omega)$ which is



$$\chi^0_{q\,\alpha\alpha'\beta\beta'}(i\Omega) = -\frac{1}{\beta}\sum_{i\omega} G_{k\,\beta\alpha}(i\omega)\,G_{k-q\,\alpha'\beta'}(i\omega - i\Omega) = \oint \frac{dz}{2\pi i} f(z)\,G_{k\,\beta\alpha}(z)\,G_{k-q\,\alpha'\beta'}(z - i\Omega)$$

$$\chi^0_q(i\Omega) = \int \frac{dx}{2\pi i}\Big[f(x)\left[G_{k\,\beta\alpha}(x+i\delta) - G_{k\,\beta\alpha}(x-i\delta)\right] G_{k-q\,\alpha'\beta'}(x - i\Omega) + f(x+i\Omega)\,G_{k\,\beta\alpha}(x+i\Omega)\left[G_{k-q\,\alpha'\beta'}(x+i\delta) - G_{k-q\,\alpha'\beta'}(x-i\delta)\right]\Big]$$

$$\chi^0_q(i\Omega) = \int \frac{dx}{\pi} f(x)\Big[\frac{1}{2i}\left(G_k(x+i\delta) - G^\dagger_k(x+i\delta)\right)_{\beta\alpha} G_{k-q\,\alpha'\beta'}(x - i\Omega) + G_{k\,\beta\alpha}(x+i\Omega)\,\frac{1}{2i}\left(G_{k-q\,\alpha'\beta'}(x+i\delta) - G^\dagger_{k-q\,\alpha'\beta'}(x+i\delta)\right)\Big]$$

$$\frac{1}{2i}\left[\chi^0_q(\Omega+i\delta) - \chi^0_q(\Omega-i\delta)\right] \equiv \chi^{0''}_q(\Omega) =$$

$$\int \frac{dx}{\pi} f(x)\Big[\frac{1}{2i}\left(G_k(x) - G^\dagger_k(x)\right)\frac{1}{2i}\left(G^\dagger_{k-q}(x-\Omega) - G_{k-q}(x-\Omega)\right) + \frac{1}{2i}\left(G_k(x+\Omega) - G^\dagger_k(x+\Omega)\right)\frac{1}{2i}\left(G_{k-q}(x) - G^\dagger_{k-q}(x)\right)\Big]$$

Define: $\rho_{k\,\alpha\beta}(x) = \frac{1}{2i}\left(G_k(x) - G^\dagger_k(x)\right)_{\alpha\beta}$

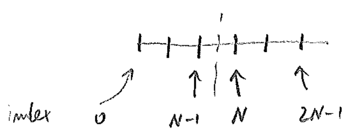
$$\boxed{\chi^{0''}_{q\,\alpha\alpha'\beta\beta'}(\Omega) = \int \frac{dx}{\pi}\left[f(x-\Omega) - f(x)\right]\rho_{k\,\beta\alpha}(x)\,\rho_{k-q\,\alpha'\beta'}(x-\Omega)}$$

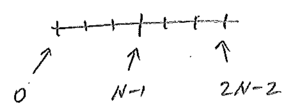
The complex form of the Kramers-Kronig relation can be used to obtain full χ^0

$$\frac{1}{2}\left[\chi(\omega+i\delta)-\chi(\omega-i\delta)\right] = -\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\frac{1}{2i}\left[\chi(x+i\delta)-\chi(x-i\delta)\right]}{\omega-x}\,dx$$

Some notes on ctqmc implementation:

fermionic frequency		bosonic frequency		
$m=0$	$\nu=-\frac{(2N-1)\pi}{\beta}$	$i\Omega=0$	$d\Omega=N_\Omega-1$	$\Omega=-2(N-1)\pi/\beta$
$\vdots$	$\vdots$	$\vdots$		
$m=N_\omega-1$	$\nu=-\frac{\pi}{\beta}$	$i\Omega=N_\Omega-1$	$d\Omega=0$	$\Omega=0$
$m=N_\omega$	$\nu=\pi/\beta$	$i\Omega=N_\Omega$	$d\Omega=1$	$\Omega=2\pi/\beta$
$\vdots$		$\vdots$		
$m=2N_\omega-1$	$\nu=\frac{(2N-1)\pi}{\beta}$	$i\Omega=2N_\Omega-2$	$d\Omega=-N_\Omega+1$	$\Omega=2(N-1)\pi/\beta$





$$\nu=\left(-(2N_\omega-1)+2m\right)\frac{\pi}{\beta} \qquad \Omega = d\Omega\,\frac{2\pi}{\beta}$$

$$m+d\Omega \xrightarrow{\text{represents}} \left(-(2N_\omega-1)+2m\right)\frac{\pi}{\beta}+d\Omega\frac{2\pi}{\beta} = \frac{\pi}{\beta}\left(-(2N_\omega-1)+2(m+d\Omega)\right)$$

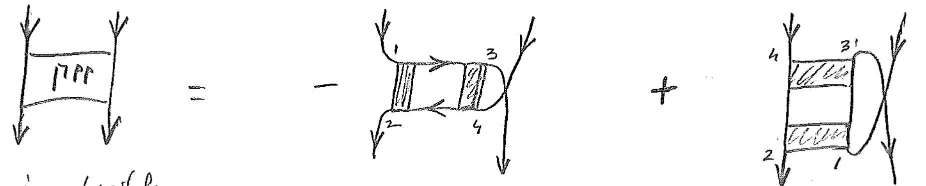
$i\Omega \in [0,\dots 2N_\Omega-1] \rightarrow \Omega = -2(N_\Omega-1)\frac{\pi}{\beta},\dots\ 2(N_\Omega-1)\frac{\pi}{\beta}$

$im_1 \in \max(0,-d\Omega),\ \min(2N_\omega, 2N_\omega-d\Omega)$

$im_2 \in \max(0,-d\Omega),\ \min(2N_\omega, 2N_\omega-d\Omega)$

Superconductivity

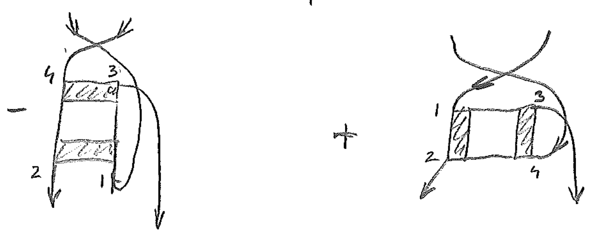
We will construct particle-particle irreducible vertex from particle hole reducible diagrams. More specifically



irreducible vertex in p-p channel

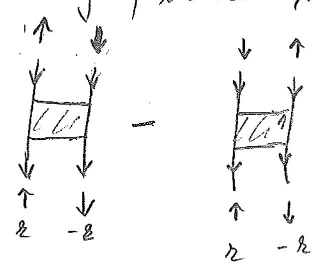
$(\Gamma\times\Gamma)^{p-h-1}$ $(\Gamma\times\Gamma)^{p-h-2}$

These two diagram can be replotted in the following equivalent form:

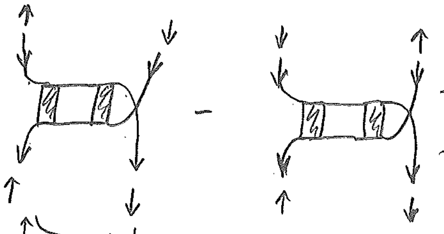


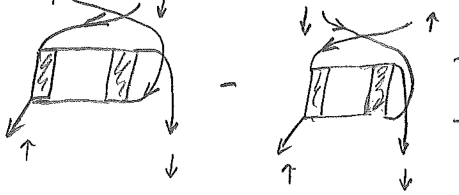
We are interested in spin-singlet pairing, hence we compute:

$\Gamma^{pp}_{singlet} = \frac{1}{2}\left(\Gamma_{\uparrow\downarrow\uparrow\downarrow} - \Gamma_{\uparrow\downarrow\downarrow\uparrow}\right)$:



We thus have:

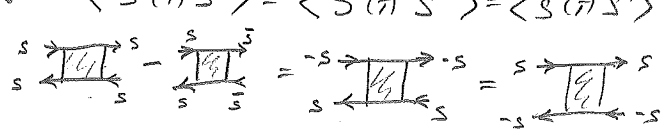
$\Gamma^{pp(1)} = -\frac{1}{2}\Big[$  $\Big] = -\frac{1}{2}\left[(\Gamma\times\Gamma)^{ph}(\uparrow\uparrow\downarrow\downarrow) - (\Gamma\times\Gamma)^{ph}(\downarrow\uparrow\uparrow\downarrow)\right]$

$\Gamma^{pp(2)} = \frac{1}{2}\Big[$  $\Big] = \frac{1}{2}\left[(\Gamma\times\Gamma)^{ph}(\downarrow\uparrow\downarrow\uparrow) - (\Gamma\times\Gamma)^{ph}(\uparrow\uparrow\downarrow\downarrow)\right]$

From definition of χ^m and χ^d it follows: $(\Gamma\times\Gamma)(\uparrow\uparrow\downarrow\downarrow) = \frac{1}{2}\left[(\Gamma\times\Gamma)^d - (\Gamma\times\Gamma)^m\right]$

In paramagnetic state we have special symmetry $\langle S^z(\tau) S^z\rangle = \langle S^+(\tau) S^-\rangle = \langle S^-(\tau) S^+\rangle$

hence

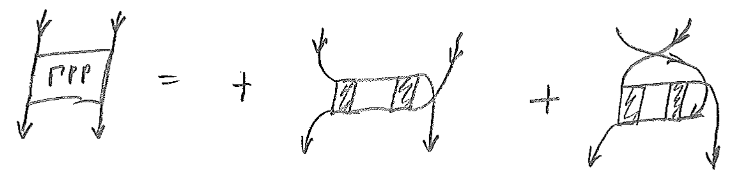


or $\chi^{ph}_{\downarrow\uparrow\downarrow\uparrow} = \chi^{ph}_{\uparrow\uparrow\uparrow\uparrow} - \chi^{ph}_{\uparrow\uparrow\downarrow\downarrow} \equiv \chi^m$

We thus have: $\Gamma^{pp(1)} = -\frac{1}{2}\left[\frac{1}{2}(\Gamma\times\Gamma)^d - \frac{1}{2}(\Gamma\times\Gamma)^m - (\Gamma\times\Gamma)^m\right] = \underline{-\frac{1}{4}(\Gamma\times\Gamma)^d + \frac{3}{4}(\Gamma\times\Gamma)^m}$

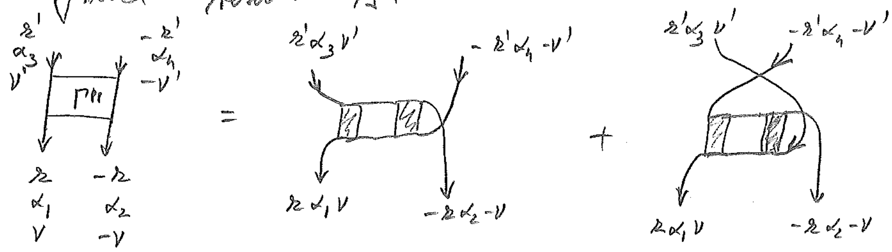
$$\Gamma^{pp(2)} = \frac{1}{2}\left[(\Gamma\times\Gamma)^m - \frac{1}{2}(\Gamma\times\Gamma)^d + \frac{1}{2}(\Gamma\times\Gamma)^m\right] = \underline{\frac{3}{4}(\Gamma\times\Gamma)^m - \frac{1}{4}(\Gamma\times\Gamma)^d}$$

We just proved that for the singlet pairing, we can drop the spin indeces and use the block $\underbrace{\frac{3}{4}(\Gamma\times\Gamma)^m - \frac{1}{4}(\Gamma\times\Gamma)^d}_{\overline{(\Gamma\times\Gamma)}}$ as the building block:



$\frac{3}{4}(\Gamma\times\Gamma)^m - \frac{1}{4}(\Gamma\times\Gamma)^d \qquad \frac{3}{4}(\Gamma\times\Gamma)^m - \frac{1}{4}(\Gamma\times\Gamma)^d$

The final result is:



$$\Gamma^{pp}(\alpha_1\alpha_2\, k\nu;\, \alpha_3\alpha_4\, k'\nu') = \overline{(\Gamma\times\Gamma)}^{ph}_{\substack{k'-k\\ \nu'-\nu}}(\alpha_3\alpha_1\nu';\, \alpha_2\alpha_4 -\nu) + \overline{(\Gamma\times\Gamma)}^{ph}_{\substack{-k'-k\\ -\nu'-\nu}}(\alpha_4\alpha_1 -\nu';\, \alpha_2\alpha_3 -\nu)$$

where $\overline{(\Gamma\times\Gamma)}^{ph} = \frac{3}{4}(\Gamma\times\Gamma)^m - \frac{1}{4}(\Gamma\times\Gamma)^d$

The Eliashberg equation reads:

$$\boxed{-\frac{1}{\beta}\sum_{\substack{k\\ \nu'\\ \alpha_3\alpha_4\alpha_5\alpha_6}} \Gamma^{pp}(\alpha_1\alpha_2\, k i\nu;\, \alpha_3\alpha_4\, k' i\nu')\, \chi^{0\,p\text{-}p}_{k'\nu'}(\alpha_3\alpha_4\, \alpha_5\alpha_6)\, \phi^{k'\nu'}_{\alpha_5\alpha_6} = \lambda\, \phi^{k\nu}_{\alpha_1\alpha_2}}$$

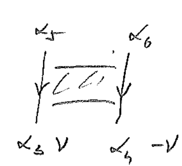
BCS approximation amounts to letting all frequency in Γ^{pp} to zero on real axis.

$\Gamma^{pp}(\alpha_1\alpha_2\, k\nu;\, \alpha_3\alpha_4\, k'\nu') \approx \Gamma^{pp}(\alpha_1\alpha_2\, k\, 0^+;\, \alpha_3\alpha_4\, k'\, 0^+)$ hence we have

$$\boxed{-\sum_{k} \Gamma^{pp}(\alpha_1\alpha_2\, k\, 0^+;\, \alpha_3\alpha_4\, k'\, 0^+)\left(\frac{1}{\beta}\sum_{i\nu'} \chi^{0\,p\text{-}p}_{i\nu'}(\alpha_3\alpha_4\, k';\, \alpha_5\alpha_6\, k')\right)\phi^{k'}_{\alpha_5\alpha_6} = \lambda\, \phi^{k}_{\alpha_1\alpha_2}}$$

where

$$\frac{1}{\beta}\sum_{i\nu} \chi^{0\,p\text{-}p}_{i\nu}(\alpha_3\alpha_4\, k;\, \alpha_5\alpha_6\, k) = \frac{1}{\beta}\sum_{i\nu} G^{k}_{\alpha_3\alpha_5}(i\nu)\, G^{-k}_{\alpha_4\alpha_6}(-i\nu) \;;$$



We are here diagonalizing

$$-\sum_{\alpha_3\alpha_4} \Gamma^{pp}(\alpha_1\alpha_2\, z\, 0^+;\ \alpha_3\alpha_4\, z'\, 0^+)\ \chi^{0\,p\text{-}p}(\alpha_3\alpha_4\, z';\ \alpha_5\alpha_6\, z') \equiv \mathcal{E}(\alpha_1\alpha_2\, z;\ \alpha_5\alpha_6\, z')$$

Here

$$\Gamma^{pp}(\alpha_1\alpha_2\, z0^+;\ \alpha_3\alpha_4\, z'0^+) = (\overline{\Gamma\chi\Gamma})^{ph}_{\substack{z'-z\\ \Omega=0}}(\alpha_3\alpha_1\, 0^+;\ \alpha_2\alpha_4\, 0^-) + (\overline{\Gamma\chi\Gamma})^{ph}_{\substack{-z'-z\\ \Omega=0}}(\alpha_4\alpha_1\, 0^-;\ \alpha_2\alpha_3\, 0^-) \quad (XY)$$

What is symmetric?

Time reversal symmetry gives: $G^{z}_{\alpha\beta}(i\omega) = G^{-z}_{\beta\alpha}(i\omega) = G^{-z\,*}_{\alpha\beta}(-i\omega)$

because $G^{z}(-i\omega) = G^{z+}(i\omega)$

We can show for bubble: 1) $\chi^0_z(\alpha_1\alpha_2;\ \alpha_3\alpha_4) = \chi^0_{-z}(\alpha_3\alpha_4;\ \alpha_1\alpha_2)$

2) $\underset{(i\nu\ i\nu'\ i\Omega)}{\chi^0_z(\alpha_1\alpha_2;\ \alpha_3\alpha_4)} = \underset{(i\nu-i\Omega,\ i\nu'-i\Omega,\ i\Omega)}{\chi^0_z(\alpha_2\alpha_1;\ \alpha_4\alpha_3)}$

Hermitian conjugate of Eq (XY) is

$$\Gamma^{pp}(\alpha_3\alpha_4\, z'0^+;\ \alpha_1\alpha_2\, z0^+)^* = (\Gamma\chi\Gamma)^{ph\,*}_{-(z'-z)}(\alpha_1\alpha_3\, 0^+;\ \alpha_4\alpha_2\, 0^-) + (\Gamma\chi\Gamma)^{ph\,*}_{-z'-z}(\alpha_2\alpha_3\, 0^-;\ \alpha_4\alpha_1\, 0^-)$$

If solution $\phi^{z} = \phi^{-z}$, we can use $\Gamma^{pp}(\alpha_1\alpha_2\, -z0^+;\ \alpha_3\alpha_4\, z'0^+)$ instead of $\Gamma^{pp}(\alpha_1\alpha_2\, z0^+;\ \alpha_3\alpha_4\, z'0^+)$

The vertex is then symmetric. But only for BCS and symmetric in z solution.