$\frac{r}{a} = \tilde{r}$ $\boxed{\text{Important}}$

$$v_r = \frac{1}{4\pi\epsilon a}\frac{e^{-\lambda r}}{\frac{r}{a}} = \frac{2}{\tilde{r}} e^{-\tilde{\lambda}\tilde{r}} \qquad v_q^\lambda = \frac{8\pi}{q^2+\lambda^2} \qquad e^2 \to 2$$

$$\Sigma_k = -\int\frac{d^3q}{(2\pi)^3} n_q v_{k-q} = -\int_0^{k_F} dq \frac{q^2}{(2\pi)^3}\int_{-1}^{1}(2\pi)dx \frac{8\pi}{\lambda^2+k^2+q^2-2kqx} = -\frac{8\pi(2\pi)}{(2\pi)^3}\int_0^{k_F} dq\, q^2 \frac{\ln\left(\frac{(q+k)^2+\lambda^2}{(q-k)^2+\lambda^2}\right)}{2kq}$$

$$\Sigma_k = -\frac{2}{\pi}\frac{1}{2k}\int_0^{k_F} dq\, q \ln\left(\frac{(k+q)^2+\lambda^2}{(k-q)^2+\lambda^2}\right) = -\frac{\lambda^2}{\pi k}\int_0^{k_F/\lambda} d\tilde{q}\,\tilde{q}\ln\left(\frac{(\tilde{k}+\tilde{q})^2+1}{(\tilde{k}-\tilde{q})^2+1}\right)$$

$\frac{k}{\lambda} = \tilde{k}$

$\frac{q}{\lambda} = \tilde{q}$

$$E_x = 2\int\frac{d^3k}{(2\pi)^3} n_k \Sigma_k = -\frac{2\lambda^2}{\pi}\frac{4\pi}{(2\pi)^3}\int_0^{k_F} dk \frac{k^2}{k}\int_0^{k_F/\lambda} d\tilde{q}\,\tilde{q}\ln\left(\frac{(\tilde{k}+\tilde{q})^2+1}{(\tilde{k}-\tilde{q})^2+1}\right) = -\frac{\lambda^4}{\pi^3}\int_0^{k_F/\lambda} d\tilde{k}\,\tilde{k}\int_0^{k_F/\lambda} d\tilde{q}\,\tilde{q}\ln\left(\frac{(\tilde{k}+\tilde{q})^2+1}{(\tilde{k}-\tilde{q})^2+1}\right)$$

$$E_x = -\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3} n_k n_q v_{q-k}$$

$$E_x = -\frac{\lambda^4}{2\pi^3}\left[\left(\frac{k_F}{\lambda}\right)^4 - \frac{1}{6}\left(\frac{k_F}{\lambda}\right)^2 - \frac{4}{3}\left(\frac{k_F}{\lambda}\right)^3 \operatorname{atan}\left(2\frac{k_F}{\lambda}\right) + \frac{1}{2}\left(\left(\frac{k_F}{\lambda}\right)^2 + \frac{1}{12}\right)\ln\left(1+4\left(\frac{k_F}{\lambda}\right)^2\right)\right]$$

$$E_x = -\frac{1}{2\pi^3} k_F^4\left[1 - \frac{1}{6}\frac{1}{x^2} - \frac{4}{3x}\operatorname{atan}(2x) + \frac{1}{2x^4}\left(x^2+\frac{1}{12}\right)\ln(1+4x^2)\right]$$

$$\rho = \frac{k_F^3}{3\pi^2} = \frac{3}{4\pi r_s^3} \qquad k_F = \left(\frac{9\pi}{4}\right)^{\frac{1}{3}}\frac{1}{r_s}$$

$$\varepsilon_x = -\frac{3\pi^2}{2\pi^3} k_F\left[1 - \frac{1}{6x^2} - \frac{4}{3x}\operatorname{atan}(2x) + \frac{1}{2x^4}\left(x^2+\frac{1}{12}\right)\ln(1+4x^2)\right] \;;\; x = \frac{k_F}{\lambda} = \frac{1}{\lambda r_s}\left(\frac{9\pi}{4}\right)^{\frac{1}{3}}$$

$$\boxed{\varepsilon_x = -\left(\frac{3}{2\pi}\right)\left(\frac{9\pi}{4}\right)^{\frac{1}{3}}\frac{1}{r_s}\left[1 - \frac{1}{6x^2} - \frac{4}{3x}\operatorname{atan}(2x) + \frac{1}{2x^2}\left(1+\frac{1}{12x^2}\right)\ln(1+4x^2)\right]} \qquad x = \left(\frac{9\pi}{4}\right)^{\frac{1}{3}}\frac{1}{(\lambda r_s)}$$

1.9

$(\lambda r_s) \to 0 \;;\; x \to \infty \qquad \varepsilon_x = \left(\frac{3}{2\pi}\right)\left(\frac{9\pi}{4}\right)^{\frac{1}{3}}\frac{1}{r_s}$

$\lambda \sim 1.5$

$r_s \gtrsim 3$

$(\lambda r_s) \to \infty \;;\; x \to 0 \qquad \varepsilon_x = -\text{//-} \times \frac{1}{r_s}\left(\frac{4}{9}x^2\right) =$

$x \lesssim \frac{1.9}{4.5} \sim 0.5$

$\varepsilon_x = -\text{//-} \times \frac{4}{9}\left(\frac{9\pi}{4}\right)^{\frac{2}{3}}\frac{1}{\lambda^2 r_s^3}$

1.64

$\underline{V_x = \frac{4}{3}\varepsilon_x}$

$\varepsilon_x = \frac{c}{r_s} \;;\; \varepsilon_x = \frac{c}{r_s}\left[\frac{c'}{(\lambda r_s)^2}\right]$

$$V_x = \frac{\delta}{\delta\rho}\int \varepsilon_x \rho \, \delta^3 r = \varepsilon_x + \rho \frac{\delta \varepsilon_x}{\delta \rho}$$

$$\varepsilon_x = \frac{c}{r_s} f(x)$$

$$V_x = \frac{4}{3}\frac{c}{r_s}\left(f(x) + \frac{1}{4} x \frac{df}{dx}\right) = \frac{4}{3}\frac{c}{r_s} f(x) + \frac{1}{3}\frac{c}{r_s} x \cdot \frac{df}{dx} = \frac{4}{3}\varepsilon_x + \frac{1}{3}\frac{c}{r_s} x \frac{df}{dx}$$

$$\frac{df}{dx} = \frac{2}{3x^3} + \frac{4}{3x^2} \operatorname{arctg}(2x) - \frac{1+6x^2}{6x^5} \ln(1+4x^2)$$

$$\frac{df}{dx} = \begin{cases} x \ll 1 & \frac{8}{9} x - \frac{32}{15} x^3 + \cdots \\ x \gg 1 & \frac{2\pi}{3x^2} - \frac{\ln 4 + 2\ln x}{x^3} + \cdots \end{cases}$$

Important

$$\boxed{\varepsilon_c^{\lambda}(r_s) = \frac{\varepsilon_c^{\lambda=0}(r_s)}{1+\sum_{m=1}^{4} Q_m r_s^m}}$$

$$\ln(1+Q_1) = \frac{\lambda(\alpha_0+\alpha_1\lambda)}{1+\alpha_2\lambda^2+\alpha_3\lambda^4+\alpha_4\lambda^6}$$

$$\ln(1+Q_2) = \frac{\lambda^2(\beta_0+\beta_1\lambda)}{1+\beta_2\lambda^2+\beta_3\lambda^4}$$

$$\ln(1+Q_3) = \frac{\lambda^3(\gamma_0+\gamma_1\lambda)}{1+\gamma_2\lambda^2}$$

$$\ln(1+Q_4) = \lambda^4(\delta_0+\delta_1\lambda^2)$$

$$V_c = \frac{\delta E_c}{\delta\rho} = \frac{\delta}{\delta\rho}\int \rho\, \varepsilon_c(\rho) = \varepsilon_c(\rho) + \rho\frac{\delta\varepsilon_c}{\delta\rho}$$

$$V_c^{\lambda} = \frac{\varepsilon_c^{\lambda=0}}{A(r_s)} + \frac{\rho}{A}\frac{\delta\varepsilon_c^{\lambda=0}}{\delta\rho} - \frac{\rho\, B(r_s)}{[A(r_s)]^2}\frac{\delta r_s}{\delta\rho}\cdot\varepsilon_c^{\lambda=0}$$

$$\boxed{V_c^{\lambda} = \frac{V_c^{\lambda=0}(r_s)}{A(r_s)} + \frac{1}{3}\frac{B(r_s)\cdot r_s}{[A(r_s)]^2}\varepsilon_c^{\lambda=0}}$$

$$\frac{\delta\rho}{\delta r_s} = -3\frac{\rho}{r_s}$$

$$\frac{\delta r_s}{\delta\rho} = -\frac{1}{3}\frac{r_s}{\rho}$$

Where: $A(r_s) = 1+\sum_{m=1}^{4} Q_m r_s^m$

$$B(r_s) = \sum_{m=1}^{4} Q_m\, m\, r_s^{m-1}$$

$$\frac{1}{C(r_s)} \equiv \frac{1}{3}\frac{B(r_s)\, r_s}{[A(r_s)]^2} = \frac{1}{3}\frac{\sum_{m=1}^{4} Q_m\, m\, r_s^m}{\left[1+\sum_{m=1}^{4} Q_m r_s^m\right]^2}$$

$$\boxed{V_c^{\lambda} = \frac{V_c^{\lambda=0}}{A(r_s)} + \frac{\varepsilon_c^{\lambda=0}}{C(r_s)}}$$

Important

$$\rho(\vec r) = \sum_{m} Y^*_{lm}(\hat r)\, u_l^2(r)\, Y_{lm'}\, M^l_{mm'} \approx \sum_m |Y_{lm}(\hat r)|^2 u_l^2(r) \left[\frac{M^l}{(2l+1)}\right] = \frac{1}{4\pi} u_l^2(r)\, M_l$$

$$\sum_{m=-l}^{l} Y^*_{lm}(\vartheta,\varphi)\, Y_{lm}(\vartheta,\varphi) = \frac{2l+1}{4\pi}$$

$$\frac{\Phi_l(r)}{r} = u_l$$

$$\boxed{\rho(r) = \frac{1}{4\pi} u_l^2(r)\, M_l} \Rightarrow \boxed{V(r)}\ ,\ \boxed{\mathcal{E}(r)}$$

$$\boxed{\rho(r) = \frac{1}{4\pi r^2} \Phi_l^2(r)\, M_l}$$

$$V_{mm'} = \langle u_l Y_{lm} | V(r) | u_l Y_{lm'} \rangle = \delta_{mm'} \int u_l^2(r)\, V(r)\, r^2 dr = \delta_{mm'} \underline{\int \Phi_l^2(r)\, V(r)\, dr}$$

<u>degenerate</u>:

$$E_{xc} = \int d^3r\, \rho(\vec r)\, \mathcal{E}_{xc}(\vec r) = \int_0^\infty \frac{u_l^2(r)}{4\pi r^2} 4\pi r^2 dr\, M_l\, \mathcal{E}_{xc}(r) = \int_0^\infty u_l^2(r)\, \mathcal{E}_{xc}(r) \times M_l$$

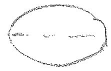
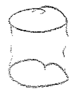
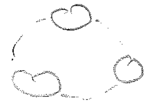
<u>non-degenerate</u>

$$\rho(\vec r) = \sum_i P_i(\vartheta,\varphi) \frac{u_l^2(r)}{r^2} M_i$$

$$E_{xc} = \underline{\sum_i} \int r^2 dr \int d\Omega\, P_i(\vartheta,\varphi) \frac{u_l^2(r)}{r^2} \underline{M_i}\, \mathcal{E}_{xc}(\vec r) = \underline{\sum_i M_i \int dr\, u_l^2(r)} \int d\Omega\, P_i(\vartheta,\varphi)\, \mathcal{E}_{xc}(\vec r)$$

$$= \int dr\, u_l^2(r) \sum_i M_i \int d\Omega\, P_i(\vartheta,\varphi)\, \mathcal{E}_{xc}(r,\vartheta,\varphi)$$

Important

$\phi = \frac{1}{2}$  $+ \frac{1}{4}$  $+ \frac{1}{6}$ 

$$\phi = -\frac{1}{V\beta}\sum_{q\, i\Omega} \frac{1}{2}\Big[\underbrace{P_q(i\Omega)\, v_q + \frac{1}{2}(P_q v_q)^2 + \frac{1}{3}(P_q v_q)^3 + \cdots}_{-\ln(1 - v_q P_q)}\Big]$$

$$\phi = \frac{1}{2}\frac{1}{V\beta}\sum_{q\, i\Omega} \ln\left(1 - v_q P_q(i\Omega)\right) = \frac{1}{2}\frac{1}{V}\oint \frac{dz}{2\pi i} M(z) \ln\left(1 - v_q P_q(z)\right) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3}\int \frac{dx}{\pi} M(x)\, \mathrm{Im}\left\{\ln\left(1 - v_q P_q(x)\right)\right.$$

$$E_{xc} = \phi - \mathrm{Tr}(\Sigma G) = \int \frac{d^3q}{(2\pi)^3}\int \frac{dx}{\pi} M(x)\, \mathrm{Im}\left\{\frac{1}{2}\ln\left(1 - v_q P_q(x)\right) + \frac{v_q P_q(x)}{1 - v_q P_q(x)}\right\}$$

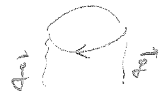
$$E_{xc} = \int_0^\infty \frac{4\pi}{8\pi^3} q^2 dq \int \frac{dx}{\pi} M(x)\, \mathrm{Im}\left\{\frac{1}{2}\ln\left(1 - v_q P_q(x)\right) + \frac{v_q P_q(x)}{1 - v_q P_q(x)}\right\}$$

$$\varepsilon_{xc} = \frac{1}{2\pi^2}\frac{3\pi^2}{k_F^3}\int_0^\infty dq\, q^2 \int_{-\infty}^{\infty} \frac{dx}{\pi} M(x)\, \mathrm{Im}\left\{\frac{1}{2}\ln\left(1 - v_q P_q(x)\right) + \frac{v_q P_q(x)}{1 - v_q P_q(x)}\right\}$$

$$\varepsilon_{xc} = \frac{3}{2k_F^3}\int_0^\infty dq\, q^2 \int_{-\infty}^{\infty} \frac{dx}{\pi} M(x)\, \mathrm{Im}\left\{\frac{1}{2}\ln\left(1 - v_q P_q(x)\right) + \frac{v_q P_q(x)}{1 - v_q P_q(x)}\right\}$$

$$\overline{v}_q = \frac{1}{\varepsilon_0}\frac{1}{q^2 + \lambda^2}$$

Important



$$P_q(i\omega) = \frac{1}{V}\frac{1}{\beta}\sum_{\substack{i\nu \\ k}} \mathcal{G}^0(k, i\nu)\,\mathcal{G}^0(k+q, i\nu+i\omega)$$

$$P_q(i\omega) = \frac{1}{V}\sum_k (-1)\oint \frac{dz}{2\pi i} f(z)\; \mathcal{G}^0(k,z)\,\mathcal{G}^0(k+q, z+i\omega)$$

$$= \frac{1}{V}\sum_k(-1)\int\frac{d\epsilon}{2\pi i}\Big\{ f(x)\left[\mathcal{G}^0(k,x+i\delta)-\mathcal{G}^0(k,x-i\delta)\right]\mathcal{G}^0(k+q,x+i\omega) +$$

$$f(x-i\omega)\,\mathcal{G}^0(k,x-i\omega)\left[\mathcal{G}^0(k+q,x+i\delta)-\mathcal{G}^0(k+q,x-i\delta)\right]\Big\}$$

$$P_q(i\omega) = \frac{1}{V}\sum_k(-1)\int\frac{d\epsilon}{\pi}\Big[f(x)\,\mathcal{G}^{0''}(k,x)\,\mathcal{G}^0(k+q,x+i\omega) + f(x)\,\mathcal{G}(k,x-i\omega)\,\mathcal{G}^{0''}(k+q,x)\Big\}$$

$$\mathrm{Im}\,P_q(\omega+i\delta) = \frac{1}{V}\sum_k(-1)\int\frac{d\epsilon}{\pi}\Big\{f(x)\,\mathcal{G}^{0''}(k,x)\,\mathcal{G}^{0''}(k+q,x+\omega) - f(x)\,\mathcal{G}^{0''}(k,x-\omega)\,\mathcal{G}^{0''}(k+q,x)\Big\}$$

$$\mathrm{Im}\,P_q(\omega+i\delta) = \frac{1}{V}\sum_k(-1)\int\frac{d\epsilon}{\pi}\left[f(x)-f(x+\omega)\right]\mathcal{G}^{0''}(k,x)\,\mathcal{G}^{0''}(k+q,x+\omega)$$

$$\mathcal{G}^0_k(\omega+) = \frac{1}{\omega+\mu-\epsilon_k+i\delta}\quad ; \quad \mathcal{G}^{0''}_k(x) = -\pi\,\delta(x+\mu-\epsilon_k)$$

$$\mathrm{Im}\,P_q(\omega+i\delta) = \frac{1}{V}\sum_k(+1)\frac{\pi}{\pi}\left[f(\epsilon_k-\mu)-f(\epsilon_k-\mu+\omega)\right]\mathcal{G}^{0''}_k(k+q,\epsilon_k-\mu+\omega)$$

$$-\epsilon_k+\epsilon_{k+q}=\omega$$

$$= \frac{1}{V}\sum_k\left[f(\epsilon_k-\mu)-f(\epsilon_k-\mu+\omega)\right](-\pi)\,\delta(\epsilon_k-\mu+\omega+\mu-\epsilon_{k+q})$$

$$\mathrm{Im}\,P_q(\omega+i\delta) = \frac{1}{V}\sum_k\left[f(\epsilon_k-\mu)-f(\epsilon_{k+q}-\mu)\right](-\pi)\,\delta(\omega+\epsilon_k-\epsilon_{k+q})$$

$$\boxed{P_q(\omega+i\delta) = \frac{1}{V}\sum_k\frac{f(\epsilon_k-\mu)-f(\epsilon_{k+q}-\mu)}{\omega+\epsilon_k-\epsilon_{k+q}+i\delta} = \sum_s\int\frac{d^3k}{(2\pi)^3}\,\frac{f(\epsilon_k-\mu)-f(\epsilon_{k+q}-\mu)}{\omega+\epsilon_k-\epsilon_{k+q}+i\delta}}$$

$$\Sigma_k(i\omega) = -\frac{1}{V}\frac{1}{\beta}\sum_{q,i\nu}\mathcal{G}^0(k+q,i\omega+i\nu)\,\frac{N_q}{1-N_q P_q(i\nu)}$$

$$\mathrm{Tr}(\Sigma G^0) = -\frac{1}{V\beta^2}\sum_{i\omega}\sum_k\sum_{q,i\Omega}\mathcal{G}^0(k+q,i\omega+i\Omega)\,\mathcal{G}_k(i\omega)\,\frac{N_q}{1-N_qP_q(i\Omega)} = -\frac{1}{V\beta}\sum_{q,i\Omega}P_q(i\Omega)\,\frac{N_q}{1-N_qP_q(i\Omega)}$$

$$= -\frac{1}{V}\sum_q\oint\frac{dz}{2\pi i}\,n(z)\,\frac{P_q(z)N_q}{1-N_qP_q(z)} = \int\frac{d^3q}{(2\pi)^3}\int\frac{d\epsilon}{\pi}\,n(x)\,\mathrm{Im}\left\{\frac{-P_q(x)N_q}{1-N_qP_q(x)}\right\}$$

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Important

$$P_q(\omega) = \sum_s \int \frac{d^3k}{(2\pi)^3} \frac{f(\xi_k - \mu) - f(\xi_{k+q} - \mu)}{\omega + \xi_k - \xi_{k+q} + i\delta} = N_s \int \frac{d^3k}{(2\pi)^3} \left[\frac{f(\xi_k - \mu)}{\omega + \xi_k - \xi_{k+q} + i\delta} - \frac{f(\xi_k - \mu)}{\omega + \xi_{k-q} - \xi_k + i\delta} \right]$$

$$= \frac{N_s}{(2\pi)^2} \int_{-1}^{1} dx \int_0^{k_F} dk\, k^2 \left[\frac{1}{\omega + \frac{k^2}{2m} - \frac{k^2 + q^2 + 2kqx}{2m} + i\delta} - \frac{1}{\omega - \frac{k^2}{2m} + \frac{k^2 + q^2 - 2kqx}{2m} + i\delta} \right]$$

$$= \frac{N_s}{(2\pi)^2} \int_0^{k_F} dk\, k^2 \int_{-1}^{1} dx \left[\frac{1}{\omega - \xi_q - \frac{kq}{m} x + i\delta} - \frac{1}{\omega + \xi_q - \frac{kq}{m} x + i\delta} \right]$$

Useful!

$$\int_{-1}^{1} \frac{dx}{e - \frac{kq}{m} x + i\delta} = \frac{m}{kq} \ln \left(\frac{e + \frac{kq}{m} + i\delta}{e - \frac{kq}{m} + i\delta} \right)$$

$$P_q(\omega) = \frac{N_s}{(2\pi)^2} \int_0^{k_F} dk\, k^2 \frac{m}{kq} \left[\ln \left(\frac{\omega - \xi_q + \frac{kq}{m}}{\omega - \xi_q - \frac{kq}{m}} \right) - \ln \left(\frac{\omega + \xi_q + \frac{kq}{m}}{\omega + \xi_q - \frac{kq}{m}} \right) \right]$$

$$= \frac{N_s}{(2\pi)^2} \frac{m}{q} \int_0^{k_F} dk\, k \left[\ln \left(\frac{\omega - \xi_q + \frac{kq}{m}}{\omega - \xi_q - \frac{kq}{m}} \right) - \ln \left(\frac{\omega + \xi_q + \frac{kq}{m}}{\omega + \xi_q - \frac{kq}{m}} \right) \right]$$

Useful!

$$\int_0^{k_F} dk\, k \left[\ln\left(e + \frac{kq}{m}\right) - \ln\left(e - \frac{kq}{m}\right) \right] = e \frac{k_F m}{q} + \frac{1}{4} \frac{2m^2}{q^2} \left(e^2 - 4 \frac{k_F^2}{2m} \frac{q^2}{2m} \right) \left[\ln\left(e - \frac{k_F q}{m}\right) - \ln\left(e + \frac{k_F q}{m}\right) \right.$$

$$P_q(\omega) = \frac{N_s}{(2\pi)^2} \frac{m}{q} \left\{ (\omega - \xi_q) \frac{k_F m}{q} + \frac{m}{4\xi_q} \left((\omega - \xi_q)^2 - 4 E_F \xi_q \right) \left[\ln\left(\omega - \xi_q - \frac{k_F}{m} q\right) - \ln\left(\omega - \xi_q + \frac{k_F}{m} q\right) \right] \right.$$

$$\left. - (\omega + \xi_q) \frac{k_F m}{q} - \frac{m}{4\xi_q} \left((\omega + \xi_q)^2 - 4 E_F \xi_q \right) \left[\ln\left(\omega + \xi_q - \frac{k_F}{m} q\right) - \ln\left(\omega + \xi_q + \frac{k_F}{m} q\right) \right] \right\}$$

$$P_q(\omega) = \frac{N_s}{(2\pi)^2} \left\{ -2\xi_q \frac{k_F m^2}{q^2} + \frac{m^2}{4q} \left(\frac{(\omega - \xi_q)^2}{\xi_q} - 4E_F \right) \left[\ln\left(\omega - \xi_q - \frac{k_F}{m} q\right) - \ln\left(\omega - \xi_q + \frac{k_F}{m} q\right) \right] \right.$$

$$\left. - \frac{m^2}{4q} \left(\frac{(\omega + \xi_q)^2}{\xi_q} - 4E_F \right) \left[\ln\left(\omega + \xi_q - \frac{k_F}{m} q\right) - \ln\left(\omega + \xi_q + \frac{k_F}{m} q\right) \right] \right\}$$

$$P_q(\omega) = \frac{N_s}{(2\pi)^2} \left\{ -k_F m + \frac{m^2}{4q} (\ldots) - \frac{m^2}{4q} (\ldots) \right\}$$

$$P_q(\omega) = -\frac{N_s}{(2\pi)^2} (m k_F) \left\{ 1 - \frac{m}{4 k_F q} \left(\frac{(\omega - \xi_q)^2}{\xi_q} - 4E_F \right) \left[\ln\left(\omega - \xi_q - \frac{k_F}{m} q\right) - \ln\left(\omega - \xi_q + \frac{k_F}{m} q\right) \right] \right.$$

$$\left. + \frac{m}{4 k_F q} \left(\frac{(\omega + \xi_q)^2}{\xi_q} - 4E_F \right) \left[\ln\left(\omega + \xi_q - \frac{k_F}{m} q\right) - \ln\left(\omega + \xi_q + \frac{k_F}{m} q\right) \right] \right\}$$

Important

Dimensionless variables

$$m k_F \to \frac{1}{Ry}\,\frac{\tilde{k}_F}{2}\,\frac{1}{a^3}$$

$$\frac{\hbar^2 q^2}{2m} \to Ry\cdot\tilde{q}^2$$

$$V_q\cdot m k_F \to \frac{8\pi}{\tilde{q}^2+\tilde{\lambda}^2}\,\frac{\tilde{k}_F}{2}$$

$$V_q \times \frac{2}{(2\pi)^2}\, m k_F \to \frac{2}{\pi}\,\frac{\tilde{k}_F}{\tilde{q}^2+\tilde{\lambda}^2}$$

$$\left(\frac{V_q}{\frac{8\pi}{q^2+\lambda^2}}\right)\times\frac{2}{(2\pi)^2}\, m k_F \to \frac{1}{4\pi^2}\,\tilde{k}_F$$

$$P_q(\omega) = -\frac{k_F}{4\pi^2}\left\{1-\frac{1}{8k_F q}\left(\frac{(\omega-q^2)^2}{q^2}-4E_F\right)\left[\ln(\omega-q^2-2k_F q)-\ln(\omega-q^2+2k_F q)\right]\right.$$
$$\left.+\frac{1}{8k_F q}\left(\frac{(\omega+q^2)^2}{q^2}-4E_F\right)\left[\ln(\omega+q^2-2k_F q)-\ln(\omega+q^2+2k_F q)\right]\right\}$$

If $q > 2k_F$

$-q(q+k_F)$ $\quad -q(q-2k_F)$ $\quad q(q-2k_F)$ $\quad q(q+2k_F)$

If $q < 2k_F$

$-\omega_p$ $\quad -q(q+2k_F)$ $\quad -q(2k_F-q)$ $\quad q(2k_F-q)$ $\quad q(q+2k_F)$ $\quad \omega_p$

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HF

Important

$$\int_0^\infty \frac{d\omega}{\pi} \rho_q''(\omega) = \begin{cases} q > 2k_F: & -\rho \\ q < 2k_F: & -\frac{q}{48\pi^2}\left(12k_F^2 - q^2\right) \end{cases}$$

$$\int_0^\infty \rho_q''(\omega)\frac{d\omega}{\pi} + \rho = -\frac{q}{48\pi^2}\left(12k_F^2 - q^2\right) + \frac{k_F^3}{3\pi^2} = \frac{1}{3\pi^2}\left[k_F^3 + \frac{q^3}{16} - \frac{3}{4} q k_F^2\right]$$

$$E_x = \frac{1}{2}\mathrm{Tr}(N_q \rho_q) = \frac{1}{2}\int \frac{d^3q}{(2\pi)^3} N_q \int_0^\infty \frac{d\omega}{\pi}\rho_q''(\omega) = \frac{1}{2}\frac{4\pi}{8\pi^3}\int dq\, q^2 \frac{8\pi}{q^2}\frac{1}{3\pi^2}\left\{k_F^3 + \frac{q^3}{16} - \frac{3}{4} q k_F^2\right\} =$$

$$= \frac{8}{3}\frac{1}{\pi}\frac{1}{2}\frac{1}{2\pi^2}\int_0^{2k_F} dq \underbrace{\left(k_F^3 + \frac{q^3}{16} - \frac{3}{4} q k_F^2\right)}_{\frac{3}{4}k_F^4} = \frac{2}{3\pi^3}\frac{3}{4}k_F^4 = \frac{1}{2}\frac{k_F^4}{\pi^3}$$

$$\varepsilon_x = \frac{E_x}{\left(\frac{k_F^3}{3\pi^2}\right)} = \frac{1}{2}\frac{k_F^4}{\pi^3}\frac{3\pi^2}{k_F^3} = \frac{3k_F}{2\pi}$$

$$\int \frac{d^3q}{(2\pi)^3} = \frac{4\pi}{8\pi^3}\int dq\, q^2 = \frac{1}{2\pi^2}\int dq\, q^2$$

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Important

$$\mathcal{E}_{xc}^{HF} = \frac{1}{2}g^2 \, Tr_{\Omega}(v_q P_q) = \frac{1}{2} g^2 \frac{8\pi}{g^2} Tr_{\Omega}(P_q) = \frac{1}{2} 8\pi \left(\sum_{k,s} n_k n_{k+q} - \sum_{k,s} n_k \right) = 4\pi \sum_{k,s} n_k n_{k+q} - 4\pi \rho$$

$$\mathcal{E}_{xc}^{HF} = 4\pi \sum_{k,s} n_k n_{k+q} - \frac{4\pi}{r_s^3}\frac{3}{4\pi} = 4\pi \sum_{k,s} n_k n_{k+q} - \frac{3}{r_s^3}$$

Here we used: $Tr_{\Omega}(P_q) = \sum_{k,s} (n_k n_{k+q} - n_k) = -\sum_{k,s} (1-n_k)\, n_{k+q}$

$$Tr_{\Omega}(P_q) = \frac{1}{\beta}\sum_{i\Omega} P_q(i\Omega) = \oint \frac{dz}{2\pi i} n(z)\, P_q(z) = \int \frac{dx}{\pi} n(x)\, P_q''(x) = \int \frac{dx}{\pi} n(x) \times N_s \int \frac{d^3k}{(2\pi)^3} (-\pi)\, \delta(x + \varepsilon_k - \varepsilon_{k+q}) [f(\varepsilon_k) - f(\varepsilon_{k+q})]$$

$$Tr_{\Omega}(P_q) = -N_s \int \frac{d^3k}{(2\pi)^3} n(\varepsilon_{k+q} - \varepsilon_k) [f(\varepsilon_k) - f(\varepsilon_{k+q})] = -N_s \int \frac{d^3k}{(2\pi)^3} f(-\varepsilon_k) f(\varepsilon_{k+q}) = -\sum_{k,s} (1-n_k)\, n_{k+q}$$

$$\boxed{Tr_{\Omega}(P_q) = \sum_{k,s} n_k n_{k+q} - \rho}$$

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Important

Problem $\quad P_q(i\Omega) = \frac{1}{\beta}\sum_{i\omega} \mathcal{G}_k(i\omega)\,\mathcal{G}_{k+q}(i\omega+i\Omega)$

1) $$\frac{1}{\beta}\sum_{i\Omega} P_q(i\Omega) = \frac{1}{\beta}\sum_{i\Omega}\frac{1}{\beta}\sum_{i\omega}\mathcal{G}_k(i\omega)\,\mathcal{G}_{k+q}(i\omega+i\Omega) = \frac{1}{\beta}\sum_{i\omega}\mathcal{G}_k(i\omega)\,\frac{1}{\beta}\sum_{i\Omega}\mathcal{G}_{k+q}(i\omega+i\Omega) =$$
$$= \underline{M_k\; M_{k+q}}$$

2) $$\frac{1}{\beta}\sum_{i\Omega} P_q(i\Omega) = \oint\frac{dz}{2\pi i}\,M(z)\,P_q(z) = \int\frac{dx}{\pi}\,M(x)\,P_q''(x)$$

$$P_q(i\Omega) = -\int\frac{dz}{2\pi i}\,f(z)\,\mathcal{G}_k(z)\,\mathcal{G}_{k+q}(z+i\Omega) = -\int\frac{dx}{\pi}\,f(x)\left[\mathcal{G}_k''(x)\,\mathcal{G}_{k+q}(x+i\Omega) + \mathcal{G}_k(x-i\Omega)\,\mathcal{G}_k''(x)\right]$$

$$P_q(\Omega+i\delta) = -\int\frac{dx}{\pi}\left[f(x)\,\mathcal{G}_k''(x)\,\mathcal{G}_{k+q}(x+\Omega) + f(x+\Omega)\,\mathcal{G}_k^*(x)\,\mathcal{G}_k''(x+\Omega)\right]$$

$$P_q''(\Omega+i\delta) = -\int\frac{dx}{\pi}\left[f(x)-f(x+\Omega)\right]\mathcal{G}_k''(x)\,\mathcal{G}_{k+q}''(x+\Omega)$$

<u>hence</u>

$$\frac{1}{\beta}\sum_{i\Omega} P_q(i\Omega) = -\int\frac{dx}{\pi}\,M(x)\int\frac{dy}{\pi}\left[f(y)-f(y+x)\right]\mathcal{G}_k''(y)\,\mathcal{G}_{k+q}''(y+x) = -\int\frac{dx}{\pi}\int\frac{dy}{\pi}\,M(x)\left[f(y)-f(y+x)\right]\times$$
$$\times\,\mathcal{G}_k''(y)\,\mathcal{G}_{k+q}''(y+x)$$

$$M(x)\left[f(y)-f(y+x)\right] = \frac{1}{e^x-1}\left[\frac{1}{e^y+1}-\frac{1}{e^{x+y}+1}\right] = \frac{e^{x+y}-e^y}{(e^x-1)(e^y+1)(e^{x+y}+1)}$$
$$= \frac{e^y\,(e^x-1)}{(e^x-1)(e^y+1)(e^{x+y}+1)} = \frac{1}{(e^{-y}+1)(e^{x+y}+1)} = f(-y)\,f(x+y)$$

$$\frac{1}{\beta}\sum_{i\Omega} P_q(i\Omega) = -\int\frac{dx}{\pi}\int\frac{dy}{\pi}\,f(-y)\,f(x+y)\,\mathcal{G}_k''(y)\,\mathcal{G}_{k+q}''(x+y) = -\underbrace{\int\frac{dy}{\pi}\,(1-f(y))\,\mathcal{G}_k''(y)}_{-(1-M_k)}\underbrace{\int\frac{dx}{\pi}\,f(x+y)\,\mathcal{G}_{k+q}''(x+y)}_{M_{k+q}}$$

$$\frac{1}{\beta}\sum_{i\Omega} P_q(i\Omega) = -(1-M_k)\,M_{k+q} = \underline{-M_{k+q} + M_k\,M_{k+q}}$$