

$$V_r = \frac{1}{4\pi\epsilon_0} \frac{e^{-\lambda r}}{r} = \frac{2}{r\lambda} e^{-\tilde{\lambda} \tilde{r}}$$

$$\frac{r}{2} = \tilde{r}$$

Important

$$l^2 \rightarrow 2$$

$$V_f = \frac{8\pi}{\tilde{r}^2 + \lambda^2}$$

$$\Sigma_2 = - \int \frac{d^3 \tilde{p}}{(2\pi)^3} M_f V_{h-f} = - \int_0^{\tilde{r}_F} d\tilde{p} \frac{\tilde{p}^2}{(2\pi)^3} \int_{-1}^1 d\lambda \frac{8\pi}{\lambda^2 + \tilde{r}^2 + \tilde{p}^2 - 2\tilde{r}\tilde{p}\lambda} = - \frac{8\pi(2\pi)}{(2\pi)^3} \int_0^{\tilde{r}_F} d\tilde{p} \tilde{p}^2 \ln \left(\frac{(\tilde{r} + \tilde{p})^2 + \lambda^2}{(\tilde{r} - \tilde{p})^2 + \lambda^2} \right)$$

$$\Sigma_2 = - \frac{2}{\pi} \frac{1}{2\tilde{r}} \int_0^{\tilde{r}_F} d\tilde{p} \tilde{p} \ln \left(\frac{(\tilde{r} + \tilde{p})^2 + \lambda^2}{(\tilde{r} - \tilde{p})^2 + \lambda^2} \right) = - \frac{\lambda^2}{\pi \tilde{r}} \int_0^{\tilde{r}_F/\lambda} d\tilde{p} \tilde{p} \ln \left(\frac{(\tilde{r} + \tilde{p})^2 + 1}{(\tilde{r} - \tilde{p})^2 + 1} \right)$$

$$\frac{\tilde{r}}{\lambda} = \tilde{z}$$

$$\frac{\tilde{p}}{\lambda} = \tilde{f}$$

$$E_x = 2 \int \frac{d^3 \tilde{p}}{(2\pi)^3} M_2 \Sigma_2 = - \frac{2\lambda^2}{\pi} \frac{4\pi}{(2\pi)^3} \int_0^{\tilde{r}_F} d\tilde{z} \frac{\tilde{z}^2}{2} \int_0^{\tilde{r}_F/\lambda} d\tilde{f} \tilde{f} \ln \left(\frac{(\tilde{z} + \tilde{f})^2 + 1}{(\tilde{z} - \tilde{f})^2 + 1} \right) = - \frac{\lambda^4}{\pi^3} \int_0^{\tilde{r}_F/\lambda} d\tilde{z} \tilde{z} \int_0^{\tilde{r}_F/\lambda} d\tilde{f} \tilde{f} \ln \left(\frac{(\tilde{z} + \tilde{f})^2 + 1}{(\tilde{z} - \tilde{f})^2 + 1} \right)$$

$$E_x = - \int \frac{d^3 \tilde{p}}{(2\pi)^3} \int \frac{d^3 \tilde{q}}{(2\pi)^3} M_2 M_f V_{f-z}$$

$$E_x = - \frac{\lambda^4}{2\pi^3} \left[\left(\frac{\tilde{r}_F}{\lambda} \right)^4 - \frac{1}{6} \left(\frac{\tilde{r}_F}{\lambda} \right)^2 - \frac{4}{3} \left(\frac{\tilde{r}_F}{\lambda} \right) \operatorname{atan}\left(2 \frac{\tilde{r}_F}{\lambda} \right) + \frac{1}{2} \left(\left(\frac{\tilde{r}_F}{\lambda} \right)^2 + \frac{1}{12} \right) \ln \left(1 + 4 \left(\frac{\tilde{r}_F}{\lambda} \right)^2 \right) \right]$$

$$E_x = - \frac{1}{2\pi^3} \tilde{r}_F^4 \left[1 - \frac{1}{6} \frac{1}{x^2} - \frac{4}{3x} \operatorname{atan}(2x) + \frac{1}{2x^2} \left(x^2 + \frac{1}{12} \right) \ln(1 + 4x^2) \right]$$

$$\tilde{r}_F = \frac{\tilde{r}_F^3}{3\pi^2} = \frac{3}{4\pi r_s^3} \quad \tilde{r}_F = \left(\frac{9\pi}{4} \right)^{\frac{1}{3}} \frac{1}{r_s}$$

$$E_x = - \frac{3\pi^2}{2\pi^3} \tilde{r}_F \left[1 - \frac{1}{6x^2} - \frac{4}{3x} \operatorname{atan}(2x) + \frac{1}{2x^2} \left(x^2 + \frac{1}{12} \right) \ln(1 + 4x^2) \right] ; x = \frac{\tilde{r}_F}{\lambda} = \frac{1}{\lambda r_s} \left(\frac{9\pi}{4} \right)^{\frac{1}{3}}$$

$$E_x = - \left(\frac{3}{2\pi} \right) \left(\frac{9\pi}{4} \right)^{\frac{1}{3}} \frac{1}{r_s} \left[1 - \frac{1}{6x^2} - \frac{4}{3x} \operatorname{atan}(2x) + \frac{1}{2x^2} \left(1 + \frac{1}{12x^2} \right) \ln(1 + 4x^2) \right] \quad x = \left(\frac{9\pi}{4} \right)^{\frac{1}{3}} \frac{1}{(\lambda r_s)}$$

$$(\lambda r_s) \rightarrow 0 \text{ if } x \rightarrow \infty$$

$$E_x = \left(\frac{3}{2\pi} \right) \left(\frac{9\pi}{4} \right)^{\frac{1}{3}} \frac{1}{r_s}$$

$$(\lambda r_s) \rightarrow \infty \text{ if } x \rightarrow 0$$

$$E_x = -11 - x \frac{1}{r_s} \left(\frac{4}{9} x^2 \right) =$$

$$E_x = -11 - x \frac{4}{9} \left(\frac{9\pi}{4} \right)^{\frac{2}{3}} \frac{1}{\lambda^2 r_s^3}$$

$$1.64$$

$$\lambda \sim 1.5$$

$$r_s \sim 3$$

$$x \leq \frac{1.9}{4.5} \sim 0.5$$

$$V_x = \frac{4}{3} E_x$$

$$E_x = \frac{c}{r_s} ; E_x = \frac{c}{r_s} \left[\frac{c}{(\lambda r_s)^2} \right]$$

$$V_x = \frac{5}{8\rho} \int \epsilon_x \rho \delta^3 r = \epsilon_x + \rho \frac{\delta \epsilon_x}{\delta \rho}$$

$$\epsilon_x = \frac{c}{2} f(x)$$

$$V_x = \frac{4}{3} \frac{c}{2} \left(f(x) + \frac{1}{4} x \frac{df}{dx} \right) = \frac{4}{3} \frac{c}{2} f(x) + \frac{1}{3} \frac{c}{2} x \cdot \frac{df}{dx} = \frac{4}{3} \epsilon_x + \frac{1}{3} \frac{c}{2} x \frac{df}{dx}$$

$$\frac{df}{dx} = \frac{2}{3x^3} + \frac{4}{3x^2} \ln(2x) - \frac{1+6x^2}{6x^5} \ln(1+4x^2)$$

$$\frac{df}{dx} = \begin{cases} x \ll 1 & \frac{8}{9}x - \frac{32}{15}x^3 + \dots \\ x \gg 1 & \frac{2\pi}{3x^2} - \frac{\ln 4 + 2 \ln x}{x^3} + \dots \end{cases}$$

Important

$$\epsilon_c^\lambda(r_s) = \frac{\epsilon_c^{\lambda=0}(r_s)}{1 + \sum_{m=1}^4 Q_m r_s^m}$$

$$\ln(1+Q_1) = \frac{\lambda(\alpha_0 + \alpha_1 \lambda)}{1 + \alpha_2 \lambda^2 + \alpha_3 \lambda^4 + \alpha_4 \lambda^6}$$

$$\ln(1+Q_2) = \frac{\lambda^2(\beta_0 + \beta_1 \lambda)}{1 + \beta_2 \lambda^2 + \beta_3 \lambda^4}$$

$$\ln(1+Q_3) = \frac{\lambda^3(\gamma_0 + \gamma_1 \lambda)}{1 + \gamma_2 \lambda^2}$$

$$\ln(1+Q_4) = \lambda^4(\delta_0 + \delta_1 \lambda^2)$$

$$V_c = \frac{\partial \epsilon_c}{\partial \rho} = \int \rho \epsilon_c(r) = \epsilon_c(r) + \rho \frac{\partial \epsilon_c}{\partial \rho}$$

$$V_c^\lambda = \frac{\epsilon_c^{\lambda=0}}{A(r_s)} + \frac{\rho}{A} \frac{\partial \epsilon_c^{\lambda=0}}{\partial \rho} - \frac{\rho B(r_s)}{[A(r_s)]^2} \frac{\partial r_s}{\partial \rho} \cdot \epsilon_c^{\lambda=0}$$

$$V_c^\lambda = \frac{V_c^{\lambda=0}}{A(r_s)} + \frac{1}{3} \frac{B(r_s) r_s}{[A(r_s)]^2} \epsilon_c^{\lambda=0}$$

$$\frac{\partial \rho}{\partial r_s} = -3 \frac{\rho}{r_s}$$

$$\frac{\partial r_s}{\partial \rho} = -\frac{1}{3} \frac{r_s}{\rho}$$

Where: $A(r_s) = 1 + \sum_{m=1}^4 Q_m r_s^m$

$$B(r_s) = \sum_{m=1}^4 Q_m m r_s^{m-1}$$

$$\frac{1}{C(r_s)} \equiv \frac{1}{3} \frac{B(r_s) r_s}{[A(r_s)]^2} = \frac{1}{3} \frac{\sum_{m=1}^4 Q_m m r_s^m}{\left[1 + \sum_{m=1}^4 Q_m r_s^m\right]^2}$$

$$V_c^\lambda = \frac{V_c^{\lambda=0}}{A(r_s)} + \frac{\epsilon_c^{\lambda=0}}{C(r_s)}$$

Important

$$\rho(\vec{r}) = \sum_m Y_{lm}^*(\vec{r}) U_l^2(r) Y_{lm} M_{mm} \approx \sum_m |Y_{lm}(\vec{r})|^2 U_l^2(r) \left[\frac{M_l^2}{l(l+1)} \right] = \frac{1}{4\pi} U_l^2(r) M_l$$

$$\sum_{m=-l}^l Y_{lm}^*(\vartheta, \varphi) Y_{lm}(\vartheta, \varphi) = \frac{2l+1}{4\pi}$$

$$\frac{\phi_l(r)}{r} = U_l$$

$$\begin{aligned} \rho(r) &= \frac{1}{4\pi} U_l^2(r) M_l \Rightarrow \boxed{V(r)} \quad \boxed{E(r)} \\ \rho(r) &= \frac{1}{4\pi r^2} \phi_l^2(r) M_l \end{aligned}$$

$$V_{mm'} = \langle U_l Y_{lm} | V(r) | U_l Y_{lm'} \rangle = \delta_{mm'} \int U_l^2(r) V(r) r^2 dr = \delta_{mm'} \int \phi_l^2(r) V(r) dr$$

degenerate:

$$E_{xc} = \int d^3r \rho(\vec{r}) E_{xc}(\vec{r}) = \int_0^\infty \frac{U_l^2(r)}{4\pi r^2} 4\pi r^2 dr M_l E_{xc}(r) = \int_0^\infty U_l^2(r) E_{xc}(r) r M_l$$

non-degenerate

$$\rho(\vec{r}) = \sum_i P_i(\vartheta, \varphi) \frac{U_l^2(r)}{r^2} M_i$$

$$\begin{aligned} E_{xc} &= \sum_i \int r^2 dr \int d\Omega P_i(\vartheta, \varphi) \frac{U_l^2(r)}{r^2} M_i E_{xc}(\vec{r}) = \sum_i M_i \int dr U_l^2(r) \int d\Omega P_i(\vartheta, \varphi) E_{xc}(\vec{r}) \\ &= \int dr U_l^2(r) \sum_i M_i \int d\Omega P_i(\vartheta, \varphi) E_{xc}(r, \vartheta, \varphi) \end{aligned}$$

$$\phi = \frac{1}{2} \text{ (circle) } + \frac{1}{4} \text{ (cylinder) } + \frac{1}{6} \text{ (sphere) }$$

Important

$$\phi = -\frac{1}{V\beta} \sum_{\vec{p} \in \Omega} \left[\frac{1}{2} \left(\vec{p} \cdot \vec{v}_{\vec{p}} \right) + \frac{1}{2} \left(\vec{p} \cdot \vec{v}_{\vec{p}} \right)^2 + \frac{1}{3} \left(\vec{p} \cdot \vec{v}_{\vec{p}} \right)^3 + \dots \right]$$

$$- \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p})$$

$$\phi = \frac{1}{2} \frac{1}{V\beta} \sum_{\vec{p} \in \Omega} \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p}) = \frac{1}{2} \frac{1}{V} \int \frac{d^3 p}{(2\pi)^3} M(\vec{p}) \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p}) = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^4 x}{\pi} M(x) \int_{\text{Im}} \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x))$$

$$E_{xc} = \phi - \text{Tr}(\Sigma G) = \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^4 x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)) + \frac{\vec{v}_{\vec{p}} \cdot \vec{p}(x)}{1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)} \right\}$$

$$E_{xc} = \int_0^\infty \frac{4\pi}{8\pi^3} p^2 dp \int \frac{d^4 x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)) + \frac{\vec{v}_{\vec{p}} \cdot \vec{p}(x)}{1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)} \right\}$$

$$E_{xc} = \frac{1}{2\pi^2} \frac{(3\pi)^2}{2\pi^3} \int_0^\infty p^2 dp \int \frac{d^4 x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)) + \frac{\vec{v}_{\vec{p}} \cdot \vec{p}(x)}{1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)} \right\}$$

$$E_{xc} = \frac{3}{2\pi^2} \int_0^\infty p^2 dp \int \frac{d^4 x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)) + \frac{\vec{v}_{\vec{p}} \cdot \vec{p}(x)}{1 - \vec{v}_{\vec{p}} \cdot \vec{p}(x)} \right\}$$

$$\vec{v}_{\vec{p}} = \frac{1}{\epsilon_0} \frac{\vec{p}}{p^2 + x^2}$$



Important

$$P_z(i\omega) = \frac{1}{V} \frac{1}{N_f} \sum_{\substack{\mathbf{k} \in \text{BZ} \\ \mathbf{k} \neq \mathbf{z}}} \psi_0^*(\mathbf{z}, i\nu) \psi_0^*(\mathbf{z} + \mathbf{g}, i\nu + i\omega)$$

$$\begin{aligned} P_z(i\omega) &= \frac{1}{V} \sum_{\mathbf{z}} (-1) \int \frac{d^2 z}{(2\pi)^2} f(\mathbf{z}) \psi_0^*(\mathbf{z}, \mathbf{z}) \psi_0^*(\mathbf{z} + \mathbf{g}, \mathbf{z} + i\omega) \\ &= \frac{1}{V} \sum_{\mathbf{z}} (-1) \int \frac{d^2 x}{(2\pi)^2} \left\{ f(\mathbf{x}) [\psi_0^*(\mathbf{z}, \mathbf{x} + i\omega) - \psi_0^*(\mathbf{z}, \mathbf{x} - i\omega)] \psi_0^*(\mathbf{z} + \mathbf{g}, \mathbf{x} + i\omega) + \right. \\ &\quad \left. f(\mathbf{x} - i\omega) \psi_0^*(\mathbf{z}, \mathbf{x} - i\omega) [\psi_0^*(\mathbf{z} + \mathbf{g}, \mathbf{x} + i\omega) - \psi_0^*(\mathbf{z} + \mathbf{g}, \mathbf{x} - i\omega)] \right\} \end{aligned}$$

$$P_z(i\omega) = \frac{1}{V} \sum_{\mathbf{z}} (-1) \int \frac{d^2 x}{\pi} \left\{ f(\mathbf{x}) \psi_0''(\mathbf{z}, \mathbf{x}) \psi_0^*(\mathbf{z} + \mathbf{g}, \mathbf{x} + i\omega) + f(\mathbf{x}) \psi_0^*(\mathbf{z}, \mathbf{x} - i\omega) \psi_0''(\mathbf{z} + \mathbf{g}, \mathbf{x}) \right\}$$

$$\lim_{\omega \rightarrow 0} P_z(i\omega + i0^+) = \frac{1}{V} \sum_{\mathbf{z}} (-1) \int \frac{d^2 x}{\pi} \left\{ f(\mathbf{x}) \psi_0''(\mathbf{z}, \mathbf{x}) \psi_0^*(\mathbf{z} + \mathbf{g}, \mathbf{x} + \omega) - f(\mathbf{x}) \psi_0^*(\mathbf{z}, \mathbf{x} - \omega) \psi_0''(\mathbf{z} + \mathbf{g}, \mathbf{x}) \right\}$$

$$\lim_{\omega \rightarrow 0} P_z(i\omega + i0^+) = \frac{1}{V} \sum_{\mathbf{z}} (-1) \int \frac{d^2 x}{\pi} [f(\mathbf{x}) - f(\mathbf{x} + \omega)] \psi_0''(\mathbf{z}, \mathbf{x}) \psi_0^*(\mathbf{z} + \mathbf{g}, \mathbf{x} + \omega)$$

$$\psi_0^*(\omega) = \frac{1}{\omega + \varepsilon - \varepsilon_{\mathbf{z} + \mathbf{g}} + i0^+}; \quad \psi_0''(\mathbf{x}) = -\pi \delta(\mathbf{x} + \mathbf{g} - \varepsilon)$$

$$\begin{aligned} \lim_{\omega \rightarrow 0} P_z(i\omega + i0^+) &= \frac{1}{V} \sum_{\mathbf{z}} (+1) \int \frac{d^2 x}{\pi} [f(\varepsilon - \mathbf{g}) - f(\varepsilon - \mathbf{g} + \omega)] \psi_0''(\mathbf{z} + \mathbf{g}, \varepsilon - \mathbf{g} + \omega) \\ &= \frac{1}{V} \sum_{\mathbf{z}} [f(\varepsilon - \mathbf{g}) - f(\varepsilon - \mathbf{g} + \omega)] (-\pi) \delta(\varepsilon - \mathbf{g} + \omega + \mathbf{g} - \varepsilon_{\mathbf{z} + \mathbf{g}}) \end{aligned}$$

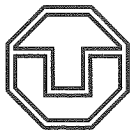
$-\varepsilon + \varepsilon_{\mathbf{z} + \mathbf{g}} = \omega$

$$\lim_{\omega \rightarrow 0} P_z(i\omega + i0^+) = \frac{1}{V} \sum_{\mathbf{z}} [f(\varepsilon - \mathbf{g}) - f(\varepsilon_{\mathbf{z} + \mathbf{g}} - \mathbf{g})] (-\pi) \delta(\omega + \varepsilon - \varepsilon_{\mathbf{z} + \mathbf{g}})$$

$$P_z(i\omega + i0^+) = \frac{1}{V} \sum_{\mathbf{z}} \frac{f(\varepsilon - \mathbf{g}) - f(\varepsilon_{\mathbf{z} + \mathbf{g}} - \mathbf{g})}{\omega + \varepsilon - \varepsilon_{\mathbf{z} + \mathbf{g}} + i0^+} = \sum_{\mathbf{z}} \int \frac{d^2 z}{(2\pi)^2} \frac{f(\varepsilon_{\mathbf{z}} - \mathbf{g}) - f(\varepsilon_{\mathbf{z} + \mathbf{g}} - \mathbf{g})}{\omega + \varepsilon - \varepsilon_{\mathbf{z} + \mathbf{g}} + i0^+}$$

$$\Sigma_{\mathbf{z}}(i\omega) = -\frac{1}{V} \frac{1}{N_f} \sum_{\substack{\mathbf{k} \in \text{BZ} \\ \mathbf{k} \neq \mathbf{z}}} \psi_0^*(\mathbf{z} + \mathbf{g}, i\omega + i\nu) \frac{N_f}{1 - N_f P_z(i\nu)}$$

$$\begin{aligned} \text{Tr}(\Sigma G) &= -\frac{1}{V} \frac{1}{N_f} \sum_{i\omega} \sum_{\mathbf{z}} \sum_{\substack{\mathbf{k} \in \text{BZ} \\ \mathbf{k} \neq \mathbf{z}}} \psi_0^*(\mathbf{z} + \mathbf{g}, i\omega + i\nu) \psi_0^*(\mathbf{z}, i\omega) \frac{N_f}{1 - N_f P_z(i\nu)} = -\frac{1}{V N_f} \sum_{\substack{\mathbf{k} \in \text{BZ} \\ \mathbf{k} \neq \mathbf{z}}} P_z(i\omega) \frac{N_f}{1 - N_f P_z(i\nu)} \\ &= -\frac{1}{V} \sum_{\mathbf{z}} \int \frac{d^2 z}{(2\pi)^2} M(\mathbf{z}) \frac{P_z(\mathbf{z}) N_f}{1 - N_f P_z(\mathbf{z})} = \int \frac{d^3 z}{(2\pi)^3} \int \frac{d^4 x}{\pi} M(\mathbf{x}) \lim_{\omega \rightarrow 0} \left\{ \frac{-P_z(\mathbf{x}) N_f}{1 - N_f P_z(\mathbf{x})} \right\} \end{aligned}$$



$$\begin{aligned}
 P_f(\omega) &= \sum_f \int \frac{d^3k}{(2\pi)^3} \frac{f(\xi_f) - f(\xi_{-f})}{\omega + \xi_f - \xi_{-f} + i0} = N_s \int \frac{d^3k}{(2\pi)^3} \left[\frac{f(\xi_f)}{\omega + \xi_f - \xi_{-f} + i0} - \frac{f(\xi_{-f})}{\omega + \xi_{-f} - \xi_f + i0} \right] \\
 &= \frac{N_s}{(2\pi)^2} \int dk \int_{-1}^1 dz k^2 \left[\frac{1}{\omega + \frac{k^2}{2m} - \frac{k^2 + y^2 + 2kyx}{2m} + i0} - \frac{1}{\omega - \frac{k^2}{2m} + \frac{k^2 + y^2 - 2kyx}{2m} + i0} \right] \\
 &= \frac{N_s}{(2\pi)^2} \int_{-1}^1 dz k^2 \int_{-1}^1 dx \left[\frac{1}{\omega - \xi_f - \frac{k_y}{m}x + i0} - \frac{1}{\omega + \xi_f - \frac{k_y}{m}x + i0} \right]
 \end{aligned}$$

Useful!

$$\int_{-1}^1 \frac{dx}{x - \frac{k_y}{m} + i0} = \frac{m}{k_y} \ln \left(\frac{0 + \frac{k_y}{m} + i0}{0 - \frac{k_y}{m} + i0} \right)$$

$$\begin{aligned}
 P_f(\omega) &= \frac{N_s}{(2\pi)^2} \int_{-1}^1 dz k^2 \frac{m}{k_y} \left[\ln \left(\frac{\omega - \xi_f + \frac{k_y}{m}}{\omega - \xi_f - \frac{k_y}{m}} \right) - \ln \left(\frac{\omega + \xi_f + \frac{k_y}{m}}{\omega + \xi_f - \frac{k_y}{m}} \right) \right] \\
 &= \frac{N_s}{(2\pi)^2} \frac{m}{2} \int_{-1}^1 dz k \left[\ln \left(\frac{\omega - \xi_f + \frac{k_y}{m}}{\omega - \xi_f - \frac{k_y}{m}} \right) - \ln \left(\frac{\omega + \xi_f + \frac{k_y}{m}}{\omega + \xi_f - \frac{k_y}{m}} \right) \right]
 \end{aligned}$$

Useful:

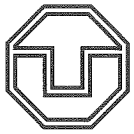
$$\int_0^{k_F} dk k \left[\ln \left(0 + \frac{k_y}{m} \right) - \ln \left(0 - \frac{k_y}{m} \right) \right] = 0 \frac{k_F m}{2} + \frac{1}{4} \frac{2m}{k_F} \left(0^2 - \frac{k_F^2}{2m} \right) \left[\ln \left(0 - \frac{k_F}{m} \right) - \ln \left(0 + \frac{k_F}{m} \right) \right]$$

$$\begin{aligned}
 P_f(\omega) &= \frac{N_s}{(2\pi)^2} \frac{m}{2} \left\{ (\omega - \xi_f) \frac{k_F m}{2} + \frac{m}{4\xi_f} (\omega - \xi_f)^2 - 4E_F \xi_f \left[\ln \left(\omega - \xi_f - \frac{k_F}{m} \right) - \ln \left(\omega - \xi_f + \frac{k_F}{m} \right) \right] \right. \\
 &\quad \left. - (\omega + \xi_f) \frac{k_F m}{2} - \frac{m}{4\xi_f} (\omega + \xi_f)^2 - 4E_F \xi_f \left[\ln \left(\omega + \xi_f - \frac{k_F}{m} \right) - \ln \left(\omega + \xi_f + \frac{k_F}{m} \right) \right] \right\}
 \end{aligned}$$

$$\begin{aligned}
 P_f(\omega) &= \frac{N_s}{(2\pi)^2} \left\{ -2\xi_f \frac{k_F m^2}{2} + \frac{m^2}{4\xi_f} \left(\frac{(\omega - \xi_f)^2}{\xi_f} - 4E_F \right) \left[\ln \left(\omega - \xi_f - \frac{k_F}{m} \right) - \ln \left(\omega - \xi_f + \frac{k_F}{m} \right) \right] \right. \\
 &\quad \left. - \frac{m^2}{4\xi_f} \left(\frac{(\omega + \xi_f)^2}{\xi_f} - 4E_F \right) \left[\ln \left(\omega + \xi_f - \frac{k_F}{m} \right) - \ln \left(\omega + \xi_f + \frac{k_F}{m} \right) \right] \right\}
 \end{aligned}$$

$$P_f(\omega) = \frac{N_s}{(2\pi)^2} \left\{ -2\xi_f m + \frac{m^2}{4\xi_f} (\dots) - \frac{m^2}{4\xi_f} (\dots) \right\}$$

$$\begin{aligned}
 P_f(\omega) &= -\frac{N_s}{(2\pi)^2} (m k_F) \left\{ 1 - \frac{m}{4\xi_f k_F} \left(\frac{(\omega - \xi_f)^2}{\xi_f} - 4E_F \right) \left[\ln \left(\omega - \xi_f - \frac{k_F}{m} \right) - \ln \left(\omega - \xi_f + \frac{k_F}{m} \right) \right] \right. \\
 &\quad \left. + \frac{m}{4\xi_f k_F} \left(\frac{(\omega + \xi_f)^2}{\xi_f} - 4E_F \right) \left[\ln \left(\omega + \xi_f - \frac{k_F}{m} \right) - \ln \left(\omega + \xi_f + \frac{k_F}{m} \right) \right] \right\}
 \end{aligned}$$



Dimensionless variables

$$m k_F \rightarrow \frac{1}{R_F} \frac{\tilde{k}_F}{2} \frac{1}{Q^3}$$

$$\frac{\hbar^2 q^2}{2m} \rightarrow R_F \cdot \tilde{q}^2$$

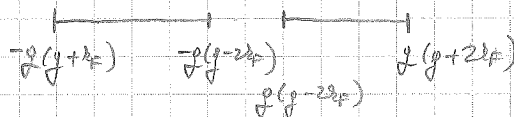
$$V_q \cdot m k_F \rightarrow \frac{\delta F}{\tilde{q}^2 + \tilde{\lambda}^2} \frac{\tilde{k}_F}{2}$$

$$V_q \times \frac{2}{(2\pi)^2} m k_F \rightarrow \frac{2}{\pi} \frac{\tilde{k}_F}{\tilde{q}^2 + \tilde{\lambda}^2}$$

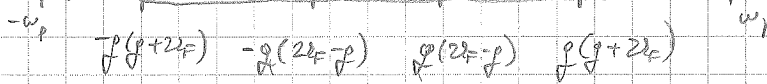
$$\left(\frac{N_q}{\frac{\delta F}{\tilde{q}^2 + \tilde{\lambda}^2}} \right) \times \frac{2}{(2\pi)^2} m k_F \rightarrow \frac{1}{4\pi^2} 2 \tilde{k}_F$$

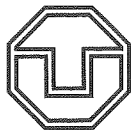
$$P_f(\omega) = - \frac{k_F}{4\pi^2} \left\{ 1 - \frac{1}{8k_F q} \left(\frac{(\omega - q^2)^2}{q^2} - 4E_F \right) \left[\ln(\omega - q^2 - 2k_F q) - \ln(\omega - q^2 + 2k_F q) \right] \right. \\ \left. + \frac{1}{8k_F q} \left(\frac{(\omega + q^2)^2}{q^2} - 4E_F \right) \left[\ln(\omega + q^2 - 2k_F q) - \ln(\omega + q^2 + 2k_F q) \right] \right\}$$

If $q > 2k_F$



If $q < 2k_F$





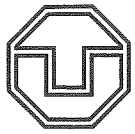
$$\int_0^{\infty} \frac{d\omega}{\pi} p_g''(\omega) = \begin{cases} g > 2\lambda_F: & -g \\ g < 2\lambda_F: & -\frac{g}{48\pi^2} (12\lambda_F^2 - g^2) \end{cases}$$

$$\int_0^{\infty} p_g''(\omega) \frac{d\omega}{\pi} + g = -\frac{g}{48\pi^2} (12\lambda_F^2 - g^2) + \frac{\lambda_F^3}{3\pi^2} = \frac{1}{3\pi^2} \left\{ \lambda_F^3 + \frac{g^3}{16} - \frac{3}{4} g \lambda_F^2 \right\}$$

$$E_x = \frac{1}{2} \text{Tr}(N_g p_g) = \frac{1}{2} \left(\frac{d^3 p}{(2\pi)^3} N_g \right) \int_0^{\infty} \frac{d\omega}{\pi} p_g''(\omega) = \frac{1}{2} \frac{4\pi}{8\pi^3} \int_0^{\lambda_F} d p \frac{8\pi}{p^2} \frac{1}{3\pi^2} \left\{ \lambda_F^3 + \frac{p^3}{16} - \frac{3}{4} p \lambda_F^2 \right\} =$$
$$= \frac{4}{3} \frac{1}{\pi} \frac{1}{2} \frac{1}{2\pi^2} \int_0^{2\lambda_F} d p \left(\lambda_F^3 + \frac{p^3}{16} - \frac{3}{4} p \lambda_F^2 \right) = \frac{2}{3} \frac{1}{\pi^3} \left(\frac{3}{4} \right) \lambda_F^4 = \frac{1}{2} \frac{\lambda_F^4}{\pi^3}$$

$$\xi_x = \frac{E_x}{\left(\frac{\lambda_F^3}{3\pi^2} \right)} = \frac{1}{2} \frac{\lambda_F^4}{\pi^3} \frac{(3\pi^2)}{\lambda_F^3} = \left(\frac{3}{2} \right) \frac{\lambda_F}{\pi}$$

$$\int \frac{d^3 p}{(2\pi)^3} = \frac{4\pi}{8\pi^3} \int_0^{\lambda_F} p^2 dp = \frac{1}{2\pi^2} \int_0^{\lambda_F} p^2 dp$$



$$E_{xc}^{HF} = \frac{1}{2} \rho^2 \text{Tr}_\rho \left(\frac{N}{\rho} P_f \right) = \frac{1}{2} \rho^2 \frac{8\pi}{\rho^2} \text{Tr}_\rho (P_f) = \frac{1}{2} 8\pi \left(\sum_{2,3} M_2 M_{2+f} - \sum_{2,2} M_2 \right) = 4\pi \sum_{2,3} M_2 M_{2+f} - 4\pi \rho$$

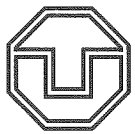
$$E_{xc}^{HF} = 4\pi \sum_{2,3} M_2 M_{2+f} - \frac{4\pi \cdot 3}{\frac{4}{3} 4\pi} = 4\pi \sum_{2,3} M_2 M_{2+f} - \frac{3}{\frac{4}{3}}$$

Here we need: $\text{Tr}_\rho(P_f) = \sum_{2,3} (M_2 M_{2+f} - M_2) = - \sum_{2,3} (1 - M_2) M_{2+f}$

$$\text{Tr}_\rho(P_f) = \frac{1}{N} \sum_{i,j} P_f(i,j) = \oint \frac{dz}{2\pi i} M(z) P_f(z) = \oint \frac{dx}{\pi} M(x) P_f''(x) = \oint \frac{dx}{\pi} M(x) \times \left(N_s \int \frac{d^3}{(2\pi)^3} (-\pi) \delta(x + \varepsilon_2 - \varepsilon_{2+f}) [f(\varepsilon_2) - f(\varepsilon_{2+f})] \right)$$

$$\text{Tr}_\rho(P_f) = -N_s \int \frac{d^3}{(2\pi)^3} M(\varepsilon_{2+f} - \varepsilon) [f(\varepsilon_2) - f(\varepsilon_{2+f})] = -N_s \int \frac{d^3}{(2\pi)^3} f(-\varepsilon_2) f(\varepsilon_{2+f}) = - \sum_{2,3} (1 - M_2) M_{2+f}$$

$$\boxed{\text{Tr}_\rho(P_f) = \sum_{2,3} M_2 M_{2+f} - \rho}$$



Problem

$$P_f(i\omega) \equiv \frac{1}{N} \sum_{i\omega} \varphi_z(i\omega) \varphi_{z+f}(i\omega + i\pi)$$

$$1) \frac{1}{N} \sum_{i\omega} P_f(i\omega) = \frac{1}{N} \sum_{i\omega} \frac{1}{N} \sum_{i\omega} \varphi_z(i\omega) \varphi_{z+f}(i\omega + i\pi) = \frac{1}{N} \sum_{i\omega} \varphi_z(i\omega) \frac{1}{N} \sum_{i\omega} \varphi_{z+f}(i\omega + i\pi) =$$

$$= \underline{\underline{M_z M_{z+f}}}$$

$$2) \frac{1}{N} \sum_{i\omega} P_f(i\omega) = \oint_{\frac{dz}{2\pi i}} M(z) P_f(z) = \oint_{\frac{dx}{2\pi i}} M(x) P_f''(x)$$

$$P_f(i\omega) = - \oint_{\frac{dz}{2\pi i}} f(z) \varphi_z(z) \varphi_{z+f}(z+i\pi) = - \oint_{\frac{dx}{2\pi i}} f(x) [\varphi_z''(x) \varphi_{z+f}(x+i\pi) + \varphi_z(x-i\pi) \varphi_z''(x)]$$

$$P_f(\omega+i\pi) = - \oint_{\frac{dx}{2\pi i}} [f(x) \varphi_z''(x) \varphi_{z+f}(x+\pi) + f(x+\pi) \varphi_z^*(x) \varphi_z''(x+\pi)]$$

$$P_f''(\omega+i\pi) = - \oint_{\frac{dx}{2\pi i}} [f(x) - f(x+\pi)] \varphi_z''(x) \varphi_{z+f}''(x+\pi)$$

hence

$$\frac{1}{N} \sum_{i\omega} P_f(i\omega) = - \oint_{\frac{dx}{2\pi i}} M(x) \oint_{\frac{dy}{2\pi i}} [f(y) - f(y+x)] \varphi_z''(y) \varphi_{z+f}''(y+x) = - \oint_{\frac{dx}{2\pi i}} \left[\oint_{\frac{dy}{2\pi i}} M(x) [f(y) - f(y+x)] \times \right.$$

$$\left. \times \varphi_z''(y) \varphi_{z+f}''(y+x) \right]$$

$$M(x) [f(y) - f(y+x)] = \frac{1}{e^x - 1} \left[\frac{1}{e^y + 1} - \frac{1}{e^{x+y} + 1} \right] = \frac{e^{x+y} - e^y}{(e^x - 1)(e^y + 1)(e^{x+y} + 1)}$$

$$= \frac{e^y (e^x - 1)}{(e^x - 1)(e^y + 1)(e^{x+y} + 1)} = \frac{1}{(e^{-y} + 1)(e^{x+y} + 1)} = f(-y) f(x+y)$$

$$\frac{1}{N} \sum_{i\omega} P_f(i\omega) = - \oint_{\frac{dx}{2\pi i}} \underbrace{\left[\oint_{\frac{dy}{2\pi i}} f(-y) f(x+y) \varphi_z''(y) \varphi_{z+f}''(x+y) \right]}_{-(1-M_z)} = - \underbrace{\left[\oint_{\frac{dx}{2\pi i}} f(x+y) \varphi_{z+f}''(x+y) \right]}_{M_{z+f}}$$

$$\frac{1}{N} \sum_{i\omega} P_f(i\omega) = - (1-M_z) M_{z+f} = - \underline{\underline{M_{z+f}}} + M_z M_{z+f}$$