

$$V_r = \frac{1}{4\pi\epsilon_0} \frac{e^{-\lambda r}}{r} = \frac{2}{r} e^{-\lambda r} \quad V_f = \frac{8\pi}{\lambda^2 + \lambda^2}$$

Important

$\lambda^2 \rightarrow 2$

$$\Sigma_2 = - \int \frac{d^3p}{(2\pi)^3} M_p V_{\text{exp}} = - \int_0^{\lambda_F} \frac{p^2}{(2\pi)^3} \int_{-1}^1 (2\pi) dx \frac{8\pi}{\lambda^2 + p^2 + p^2 - 2ipx} = - \frac{8\pi (2\pi)}{(2\pi)^3} \int_0^{\lambda_F} dp p^2 \ln \left( \frac{(p+\lambda)^2 + \lambda^2}{(p-\lambda)^2 + \lambda^2} \right)$$

$$\Sigma_2 = - \frac{2}{\pi} \frac{1}{2\lambda} \int_0^{\lambda_F} dp p \ln \left( \frac{(p+\lambda)^2 + \lambda^2}{(p-\lambda)^2 + \lambda^2} \right) = - \frac{\lambda}{\pi} \int_0^{\lambda_F/\lambda} d\tilde{p} \tilde{p} \ln \left( \frac{(\tilde{p}+1)^2 + 1}{(\tilde{p}-1)^2 + 1} \right)$$

$$\frac{\lambda}{\lambda} = \tilde{\lambda}$$

$$\frac{p}{\lambda} = \tilde{p}$$

$$E_x = 2 \int \frac{p^2}{(2\pi)^3} M_e \Sigma_2 = - \frac{2\lambda^2}{\pi} \frac{4\pi}{(2\pi)^3} \int_0^{\lambda_F} d\tilde{p} \tilde{p} \int_0^{\lambda_F/\lambda} d\tilde{p} \tilde{p} \ln \left( \frac{(\tilde{p}+1)^2 + 1}{(\tilde{p}-1)^2 + 1} \right) = - \frac{\lambda^4}{\pi^3} \int_0^{\lambda_F/\lambda} d\tilde{p} \tilde{p} \int_0^{\lambda_F/\lambda} d\tilde{p} \tilde{p} \ln \left( \frac{(\tilde{p}+1)^2 + 1}{(\tilde{p}-1)^2 + 1} \right)$$

$$E_x = - \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3p}{(2\pi)^3} M_e M_p V_{p-z}$$

$$E_x = - \frac{\lambda^4}{2\pi^3} \left[ \left(\frac{\lambda_F}{\lambda}\right)^4 - \frac{1}{6} \left(\frac{\lambda_F}{\lambda}\right)^2 - \frac{4}{3} \left(\frac{\lambda_F}{\lambda}\right) \operatorname{atan}\left(2\frac{\lambda_F}{\lambda}\right) + \frac{1}{2} \left( \left(\frac{\lambda_F}{\lambda}\right)^2 + \frac{1}{12} \right) \ln \left( 1 + 4 \left(\frac{\lambda_F}{\lambda}\right)^2 \right) \right]$$

$$E_x = - \frac{1}{2\pi^3} \lambda_F^4 \left[ 1 - \frac{1}{6} \frac{1}{x^2} - \frac{4}{3x} \operatorname{atan}(2x) + \frac{1}{2x^2} \left( x^2 + \frac{1}{12} \right) \ln(1 + 4x^2) \right]$$

$$g = \frac{\lambda_F^3}{3\pi^2} = \frac{3}{4\pi r_s^3} \quad \lambda_F = \left(\frac{9\pi}{4}\right)^{\frac{1}{3}} \frac{1}{r_s}$$

$$E_x = - \frac{3\pi^2}{2\pi^3} \lambda_F \left[ 1 - \frac{1}{6x^2} - \frac{4}{3x} \operatorname{atan}(2x) + \frac{1}{2x^2} \left( x^2 + \frac{1}{12} \right) \ln(1 + 4x^2) \right] ; x = \frac{\lambda_F}{\lambda} = \frac{1}{\lambda r_s} \left(\frac{9\pi}{4}\right)^{\frac{1}{3}}$$

$$E_x = - \left(\frac{3}{2\pi}\right) \left(\frac{9\pi}{4}\right)^{\frac{1}{3}} \frac{1}{r_s} \left[ 1 - \frac{1}{6x^2} - \frac{4}{3x} \operatorname{atan}(2x) + \frac{1}{2x^2} \left( 1 + \frac{1}{12x^2} \right) \ln(1 + 4x^2) \right] \quad x = \left(\frac{9\pi}{4}\right)^{\frac{1}{3}} \frac{1}{\lambda r_s}$$

$$(\lambda r_s) \rightarrow 0 \text{ of } x \rightarrow \infty \quad E_x = \left(\frac{3}{2\pi}\right) \left(\frac{9\pi}{4}\right)^{\frac{1}{3}} \frac{1}{r_s}$$

$$(\lambda r_s) \rightarrow \infty \text{ of } x \rightarrow 0 \quad E_x = -11 - x \frac{1}{r_s} \left(\frac{4}{9} x^2\right) =$$

$$E_x = -11 - x \frac{4}{9} \left(\frac{9\pi}{4}\right)^{\frac{2}{3}} \frac{1}{\lambda r_s^3}$$

$\lambda \sim 1.5$

$r_s \geq 5$

$x \leq \frac{1.9}{4.5} \sim 0.5$

$$V_x = \frac{4}{3} E_x$$

1.64

$$E_x = \frac{c}{r_s} ; E_x = \frac{c}{r_s} \left[ \frac{c}{(\lambda r_s)^2} \right]$$

$$V_x = \int \frac{\delta}{\delta p} \int \epsilon_x \rho \delta^3 r = \epsilon_x + \rho \frac{\delta \epsilon_x}{\delta p}$$

$$\epsilon_x = \frac{c}{v} f(x)$$

$$V_x = \frac{4}{3} \frac{c}{v} \left( f(x) + \frac{1}{4} x \frac{df}{dx} \right) = \frac{4}{3} \frac{c}{v} f(x) + \frac{1}{3} \frac{c}{v} x \cdot \frac{df}{dx} = \frac{4}{3} \epsilon_x + \frac{1}{3} \frac{c}{v} x \frac{df}{dx}$$

$$\frac{df}{dx} = \frac{2}{3x^3} + \frac{4}{3x^2} \operatorname{arctan}(2x) - \frac{1+6x^2}{6x^4} \ln(1+4x^2)$$

$$\frac{df}{dx} = \begin{cases} x \ll 1 & \frac{8}{9} x - \frac{32}{15} x^3 + \dots \\ x \gg 1 & \frac{2\pi}{3x^2} - \frac{\ln 4 + 2 \ln x}{x^3} + \dots \end{cases}$$

Important

$$E_c^\lambda(r_s) = \frac{E_c^{\lambda=0}(r_s)}{1 + \sum_{m=1}^4 \alpha_m r_s^m}$$

$$\ln(1 + \alpha_1) = \frac{\lambda(\alpha_0 + \alpha_1 \lambda)}{1 + \alpha_2 \lambda^2 + \alpha_3 \lambda^4 + \alpha_4 \lambda^6}$$

$$\ln(1 + \alpha_2) = \frac{\lambda^2(\beta_0 + \beta_1 \lambda)}{1 + \beta_2 \lambda^2 + \beta_3 \lambda^4}$$

$$\ln(1 + \alpha_3) = \frac{\lambda^3(\gamma_0 + \gamma_1 \lambda)}{1 + \gamma_2 \lambda^2}$$

$$\ln(1 + \alpha_4) = \lambda^4(\delta_0 + \delta_1 \lambda^2)$$

$$V_c = \frac{\partial E_c}{\partial p} = \int p E_c(p) = E_c(p) + p \frac{\partial E_c}{\partial p}$$

$$V_c^\lambda = \frac{E_c^{\lambda=0}}{A(r_s)} + \frac{p}{A} \frac{\partial E_c^{\lambda=0}}{\partial p} - \frac{p B(r_s)}{[A(r_s)]^2} \frac{\partial r_s}{\partial p} \cdot E_c^{\lambda=0}$$

$$V_c^\lambda = \frac{V_c^{\lambda=0}}{A(r_s)} + \frac{1}{3} \frac{B(r_s) r_s}{[A(r_s)]^2} E_c^{\lambda=0}$$

$$\frac{\partial p}{\partial r_s} = -3 \frac{p}{r_s}$$

$$\frac{\partial r_s}{\partial p} = -\frac{1}{3} \frac{r_s}{p}$$

Where:  $A(r_s) = 1 + \sum_{m=1}^4 \alpha_m r_s^m$

$$B(r_s) = \sum_{m=1}^4 \alpha_m m r_s^{m-1}$$

$$\frac{1}{C(r_s)} \equiv \frac{1}{3} \frac{B(r_s) r_s}{[A(r_s)]^2} = \frac{1}{3} \frac{\sum_{m=1}^4 \alpha_m m r_s^m}{\left[1 + \sum_{m=1}^4 \alpha_m r_s^m\right]^2}$$

$$V_c^\lambda = \frac{V_c^{\lambda=0}}{A(r_s)} + \frac{E_c^{\lambda=0}}{C(r_s)}$$

Important

$$\rho(\vec{r}) = \sum_m Y_{lm}^*(\vec{r}) U_l^2(r) Y_{lm} M_{mm} \approx \sum_m |Y_{lm}(\vec{r})|^2 U_l^2(r) \left[ \frac{M_l}{(2l+1)} \right] = \frac{1}{4\pi} U_l^2(r) M_l$$

$$\sum_{m=-l}^l Y_{lm}^*(\vartheta, \varphi) Y_{lm}(\vartheta, \varphi) = \frac{2l+1}{4\pi}$$

$$\frac{\phi_l^2(r)}{r} = M_l$$

$$\rho(r) = \frac{1}{4\pi} U_l^2(r) M_l \Rightarrow \boxed{V(r)} \quad \boxed{E(r)}$$

$$\rho(r) = \frac{1}{4\pi r^2} \phi_l^2(r) M_l$$

$$V_{mm'} = \langle M_l Y_{lm} | V(r) | M_l Y_{lm} \rangle = \delta_{mm'} \int U_l^2(r) V(r) r^2 dr = \delta_{mm'} \int \phi_l^2(r) V(r) dr$$

degenerate:

$$\bar{E}_{xc} = \int d^3r \rho(\vec{r}) E_{xc}(\vec{r}) = \int_0^\infty \frac{U_l^2(r)}{4\pi r^2} 4\pi r^2 dr M_l E_{xc}(r) = \int_0^\infty U_l^2(r) E_{xc}(r) r dr M_l$$

non-degenerate

$$\rho(\vec{r}) = \sum_i P_i(\vartheta, \varphi) \frac{U_l^2(r)}{r^2} M_i$$

$$E_{xc} = \sum_i \int r^2 dr \int d\Omega P_i(\vartheta, \varphi) \frac{U_l^2(r)}{r^2} M_i E_{xc}(\vec{r}) = \sum_i M_i \int dr U_l^2(r) \int d\Omega P_i(\vartheta, \varphi) E_{xc}(\vec{r})$$

$$= \int dr U_l^2(r) \sum_i M_i \int d\Omega P_i(\vartheta, \varphi) E_{xc}(r, \vartheta, \varphi)$$

Ympetant

$$\Phi = \frac{1}{2} \text{---} + \frac{1}{4} \text{---} + \frac{1}{6} \text{---}$$

$$\Phi = -\frac{1}{V\Omega} \sum_{p \in \Omega} \left[ \frac{1}{2} \left( \frac{P_p(z)}{f} \right) \frac{N_p}{f} + \frac{1}{2} \left( \frac{P_p}{f} \frac{N_p}{f} \right)^2 + \frac{1}{3} \left( \frac{P_p}{f} \frac{N_p}{f} \right)^3 + \dots \right] - \ln(1 - \frac{N_p P_p}{f})$$

$$\Phi = \frac{1}{2} \frac{1}{V\Omega} \sum_{p \in \Omega} \ln(1 - \frac{N_p P_p(z)}{f}) = \frac{1}{2} \frac{1}{V} \int \frac{d^3z}{(2\pi)^3} M(z) \ln(1 - \frac{N_p P_p(z)}{f}) = \frac{1}{2} \int \frac{d^3z}{(2\pi)^3} \int \frac{d^4x}{\pi} M(x) \int_{\text{Im}} \left\{ \ln(1 - \frac{N_p P_p(x)}{f}) \right\}$$

$$E_{xc} = \Phi - \text{Tr}(\Sigma G) = \int \frac{d^3z}{(2\pi)^3} \int \frac{d^4x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \frac{N_p P_p(x)}{f}) + \frac{N_p P_p(x)}{1 - N_p P_p(x)} \right\}$$

$$E_{xc} = \int_0^{\infty} \frac{4\pi}{8\pi^3} \int_{\text{Im}} \int_{\text{Im}} \int \frac{d^4x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \frac{N_p P_p(x)}{f}) + \frac{N_p P_p(x)}{1 - N_p P_p(x)} \right\}$$

$$E_{xc} = \frac{1}{2} \frac{(3V)^2}{2f^3} \int_0^{\infty} \int_{\text{Im}} \int_{\text{Im}} \int \frac{d^4x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \frac{N_p P_p(x)}{f}) + \frac{N_p P_p(x)}{1 - N_p P_p(x)} \right\}$$

$$E_{xc} = \frac{3}{2} \frac{V^2}{2f^3} \int_0^{\infty} \int_{\text{Im}} \int_{\text{Im}} \int \frac{d^4x}{\pi} M(x) \int_{\text{Im}} \left\{ \frac{1}{2} \ln(1 - \frac{N_p P_p(x)}{f}) + \frac{N_p P_p(x)}{1 - N_p P_p(x)} \right\}$$

$$\frac{N_p}{f} = \frac{1}{\epsilon_0} \frac{1}{f^2 + x^2}$$



Important

$$P_f(i\omega) = \frac{1}{\sqrt{13}} \sum_{\substack{z \\ \Re z = 2}} \psi_f^\circ(z, i\omega) \psi_f^\circ(z + \epsilon_f, i\omega + i\epsilon_f)$$

$$\begin{aligned} P_f(i\omega) &= \frac{1}{\sqrt{13}} \int_{\Sigma} (-1) \left( \frac{dz}{2\pi i} \right) f(z) \psi_f^\circ(z, z) \psi_f^\circ(z + \epsilon_f, z + i\omega) \\ &= \frac{1}{\sqrt{13}} \sum_{\Sigma} (-1) \left( \frac{dx}{2\pi i} \right) \left\{ f(x) [\psi_f^\circ(z, x + i\epsilon_f) - \psi_f^\circ(z, x - i\epsilon_f)] \psi_f^\circ(z + \epsilon_f, x + i\omega) + \right. \\ &\quad \left. f(x - i\omega) \psi_f^\circ(z, x - i\omega) [\psi_f^\circ(z + \epsilon_f, x + i\epsilon_f) - \psi_f^\circ(z + \epsilon_f, x - i\epsilon_f)] \right\} \end{aligned}$$

$$P_f(i\omega) = \frac{1}{\sqrt{13}} \sum_{\Sigma} (-1) \int \frac{dx}{\pi} \left\{ f(x) \psi_f^{\circ\prime\prime}(z, x) \psi_f^\circ(z + \epsilon_f, x + i\omega) + f(x) \psi_f^\circ(z, x - i\omega) \psi_f^{\circ\prime\prime}(z + \epsilon_f, x) \right\}$$

$$\Im P_f(\omega + i\epsilon_f) = \frac{1}{\sqrt{13}} \sum_{\Sigma} (-1) \int \frac{dx}{\pi} \left\{ f(x) \psi_f^{\circ\prime\prime}(z, x) \psi_f^{\circ\prime\prime}(z + \epsilon_f, x + \omega) - f(x) \psi_f^{\circ\prime\prime}(z, x - \omega) \psi_f^{\circ\prime\prime}(z + \epsilon_f, x) \right\}$$

$$\Im P_f(\omega + i\epsilon_f) = \frac{1}{\sqrt{13}} \sum_{\Sigma} (-1) \int \frac{dx}{\pi} [f(x) - f(x + \omega)] \psi_f^{\circ\prime\prime}(z, x) \psi_f^{\circ\prime\prime}(z + \epsilon_f, x + \omega)$$

$$\psi_f^\circ(z) = \frac{1}{\omega + \epsilon_f - \epsilon_f + i\epsilon_f}; \quad \psi_f^{\circ\prime\prime}(x) = -\pi \delta(x + \epsilon_f - \epsilon_f)$$

$$\begin{aligned} \Im P_f(\omega + i\epsilon_f) &= \frac{1}{\sqrt{13}} \sum_{\Sigma} (+1) \int \frac{dx}{\pi} [f(\epsilon_f - \epsilon_f) - f(\epsilon_f - \epsilon_f + \omega)] \psi_f^{\circ\prime\prime}(z + \epsilon_f, \epsilon_f - \epsilon_f + \omega) \\ &= \frac{1}{\sqrt{13}} \sum_{\Sigma} [f(\epsilon_f - \epsilon_f) - f(\epsilon_f - \epsilon_f + \omega)] (-\pi) \delta(\epsilon_f - \epsilon_f + \omega + \epsilon_f - \epsilon_f) \end{aligned}$$

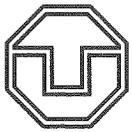
$$-\epsilon_f + \epsilon_f = \omega$$

$$\Im P_f(\omega + i\epsilon_f) = \frac{1}{\sqrt{13}} \sum_{\Sigma} [f(\epsilon_f - \epsilon_f) - f(\epsilon_f + \epsilon_f - \epsilon_f)] (-\pi) \delta(\omega + \epsilon_f - \epsilon_f)$$

$$P_f(\omega + i\epsilon_f) = \frac{1}{\sqrt{13}} \sum_{\Sigma} \frac{f(\epsilon_f - \epsilon_f) - f(\epsilon_f + \epsilon_f - \epsilon_f)}{\omega + \epsilon_f - \epsilon_f + i\epsilon_f} = \int \frac{dz}{2\pi} \frac{f(\epsilon_f - \epsilon_f) - f(\epsilon_f + \epsilon_f - \epsilon_f)}{\omega + \epsilon_f - \epsilon_f + i\epsilon_f}$$

$$\Sigma_2(i\omega) = -\frac{1}{\sqrt{13}} \sum_{\substack{z \\ \Re z = 2}} \psi_f^\circ(z + \epsilon_f, i\omega + i\epsilon_f) \frac{N_f}{1 - N_f P_f(i\omega)}$$

$$\begin{aligned} \text{TR}(\Sigma_2) &= -\frac{1}{\sqrt{13}} \sum_{i\omega} \sum_{\Sigma} \sum_{\substack{z \\ \Re z = 2}} \psi_f^\circ(z + \epsilon_f, i\omega + i\epsilon_f) \psi_f^\circ(z, i\omega) \frac{N_f}{1 - N_f P_f(i\omega)} = -\frac{1}{\sqrt{13}} \sum_{\substack{z \\ \Re z = 2}} P_f(i\omega) \frac{N_f}{1 - N_f P_f(i\omega)} \\ &= -\frac{1}{\sqrt{13}} \int \frac{dz}{2\pi} M(z) \frac{P_f(z) N_f}{1 - N_f P_f(z)} = \int \frac{dz}{2\pi} \int \frac{dx}{\pi} M(x) \Im \left\{ \frac{-P_f(x) N_f}{1 - N_f P_f(x)} \right\} \end{aligned}$$



$$P_f(\omega) = \sum_s \int_{-\infty}^{\infty} \frac{d^2 z}{(2\pi)^2} \frac{f(\xi_f - z) - f(\xi_f + z)}{\omega + \xi_f - \xi_f + i\delta} = N_s \left( \frac{d^2 z}{(2\pi)^2} \left[ \frac{f(\xi_f)}{\omega + \xi_f - \xi_f + i\delta} - \frac{f(\xi_f)}{\omega + \xi_f - \xi_f + i\delta} \right] \right)$$

$$= \frac{N_s}{(2\pi)^2} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' z' \left[ \frac{1}{\omega + \frac{z^2}{2m} - \frac{z'^2 + y^2 + 2zyx}{2m} + i\delta} - \frac{1}{\omega - \frac{z^2}{2m} + \frac{z'^2 + y^2 - 2zyx}{2m} + i\delta} \right]$$

$$= \frac{N_s}{(2\pi)^2} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' z' \left[ \frac{1}{\omega - \xi_f - \frac{2y}{m} x + i\delta} - \frac{1}{\omega + \xi_f - \frac{2y}{m} x + i\delta} \right]$$

Useful!

$$\int_{-\infty}^{\infty} \frac{dx}{x - \frac{2y}{m} x + i\delta} = \frac{m}{2y} \ln \left( \frac{\omega + \frac{2y}{m} + i\delta}{\omega - \frac{2y}{m} + i\delta} \right)$$

$$P_f(\omega) = \frac{N_s}{(2\pi)^2} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' z' \frac{m}{2y} \left[ \ln \left( \frac{\omega - \xi_f + \frac{2y}{m}}{\omega - \xi_f - \frac{2y}{m}} \right) - \ln \left( \frac{\omega + \xi_f + \frac{2y}{m}}{\omega + \xi_f - \frac{2y}{m}} \right) \right]$$

$$= \frac{N_s}{(2\pi)^2} \frac{m}{2y} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' z' \left[ \ln \left( \frac{\omega - \xi_f + \frac{2y}{m}}{\omega - \xi_f - \frac{2y}{m}} \right) - \ln \left( \frac{\omega + \xi_f + \frac{2y}{m}}{\omega + \xi_f - \frac{2y}{m}} \right) \right]$$

Useful!

$$\int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' z' \left[ \ln \left( \frac{\omega + \frac{2y}{m}}{\omega - \frac{2y}{m}} \right) - \ln \left( \frac{\omega + \frac{2y}{m}}{\omega - \frac{2y}{m}} \right) \right] = \frac{2y m}{2} + \frac{1}{4} \frac{2m}{y^2} \left( \omega^2 - 4 \frac{2y^2}{2m} \frac{y^2}{2m} \right) \left[ \ln \left( \frac{\omega + \frac{2y}{m}}{\omega - \frac{2y}{m}} \right) - \ln \left( \frac{\omega + \frac{2y}{m}}{\omega - \frac{2y}{m}} \right) \right]$$

$$P_f(\omega) = \frac{N_s}{(2\pi)^2} \frac{m^2}{2} \left\{ \left( \frac{\omega - \xi_f}{\xi_f} \right)^2 \frac{2y m}{2} + \frac{m}{4 \xi_f} \left( \frac{\omega - \xi_f}{\xi_f} \right)^2 - 4 E_F \xi_f \left[ \ln \left( \frac{\omega - \xi_f - \frac{2y}{m}}{\omega - \xi_f + \frac{2y}{m}} \right) - \ln \left( \frac{\omega - \xi_f + \frac{2y}{m}}{\omega - \xi_f - \frac{2y}{m}} \right) \right] \right.$$

$$\left. - \left( \frac{\omega + \xi_f}{\xi_f} \right)^2 \frac{2y m}{2} - \frac{m}{4 \xi_f} \left( \frac{\omega + \xi_f}{\xi_f} \right)^2 - 4 E_F \xi_f \left[ \ln \left( \frac{\omega + \xi_f - \frac{2y}{m}}{\omega + \xi_f + \frac{2y}{m}} \right) - \ln \left( \frac{\omega + \xi_f + \frac{2y}{m}}{\omega + \xi_f - \frac{2y}{m}} \right) \right] \right\}$$

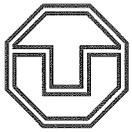
$$P_f(\omega) = \frac{N_s}{(2\pi)^2} \left\{ -2 \xi_f \frac{2y m^2}{2} + \frac{m^2}{4 \xi_f} \left( \frac{\omega - \xi_f}{\xi_f} \right)^2 - 4 E_F \xi_f \left[ \ln \left( \frac{\omega - \xi_f - \frac{2y}{m}}{\omega - \xi_f + \frac{2y}{m}} \right) - \ln \left( \frac{\omega - \xi_f + \frac{2y}{m}}{\omega - \xi_f - \frac{2y}{m}} \right) \right] \right.$$

$$\left. - \frac{m^2}{4 \xi_f} \left( \frac{\omega + \xi_f}{\xi_f} \right)^2 - 4 E_F \xi_f \left[ \ln \left( \frac{\omega + \xi_f - \frac{2y}{m}}{\omega + \xi_f + \frac{2y}{m}} \right) - \ln \left( \frac{\omega + \xi_f + \frac{2y}{m}}{\omega + \xi_f - \frac{2y}{m}} \right) \right] \right\}$$

$$P_f(\omega) = \frac{N_s}{(2\pi)^2} \left\{ -2 \xi_f m + \frac{m^2}{4 \xi_f} (\dots) - \frac{m^2}{4 \xi_f} (\dots) \right\}$$

$$P_f(\omega) = -\frac{N_s}{(2\pi)^2} (m 2 \xi_f) \left\{ 1 - \frac{m}{4 \xi_f} \left( \frac{\omega - \xi_f}{\xi_f} \right)^2 - 4 E_F \xi_f \left[ \ln \left( \frac{\omega - \xi_f - \frac{2y}{m}}{\omega - \xi_f + \frac{2y}{m}} \right) - \ln \left( \frac{\omega - \xi_f + \frac{2y}{m}}{\omega - \xi_f - \frac{2y}{m}} \right) \right] \right.$$

$$\left. + \frac{m}{4 \xi_f} \left( \frac{\omega + \xi_f}{\xi_f} \right)^2 - 4 E_F \xi_f \left[ \ln \left( \frac{\omega + \xi_f - \frac{2y}{m}}{\omega + \xi_f + \frac{2y}{m}} \right) - \ln \left( \frac{\omega + \xi_f + \frac{2y}{m}}{\omega + \xi_f - \frac{2y}{m}} \right) \right] \right\}$$



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Dimensionless variables

$$M \kappa_F \rightarrow \frac{1}{R_F} \frac{\tilde{\kappa}_F}{2} \frac{1}{Q^3}$$

$$\frac{\hbar^2 q^2}{2m} \rightarrow R_F \cdot \tilde{q}^2$$

$$V \cdot M \kappa_F \rightarrow \frac{\delta F}{\tilde{q}^2 + \tilde{\chi}^2} \frac{\tilde{\kappa}_F}{2}$$

$$N_q \times \frac{2}{(2\pi)^2} M \kappa_F \rightarrow \frac{2}{\pi} \frac{\tilde{\kappa}_F}{\tilde{q}^2 + \tilde{\chi}^2}$$

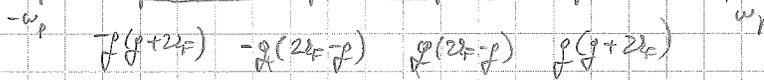
$$\left( \frac{N_q}{\tilde{q}^2 + \tilde{\chi}^2} \right) \times \frac{2}{(2\pi)^2} M \kappa_F \rightarrow \frac{1}{4\pi^2} \tilde{\kappa}_F$$

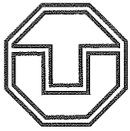
$$P_f(\omega) = - \frac{\kappa_F}{4\pi^2} \left\{ 1 - \frac{1}{8\kappa_F q} \left( \frac{(\omega - q^2)^2}{q^2} - 4E_F \right) \left[ \ln(\omega - q^2 - 2\kappa_F q) - \ln(\omega - q^2 + 2\kappa_F q) \right] \right. \\ \left. + \frac{1}{8\kappa_F q} \left( \frac{(\omega + q^2)^2}{q^2} - 4E_F \right) \left[ \ln(\omega + q^2 - 2\kappa_F q) - \ln(\omega + q^2 + 2\kappa_F q) \right] \right\}$$

If  $q > 2\kappa_F$



If  $q < 2\kappa_F$





#F

$$\int_0^{\infty} \frac{d\omega}{\pi} P_g''(\omega) = \begin{cases} g > 2z_F: & -g \\ g < 2z_F: & -\frac{g}{48\pi^2} (12z_F^2 - g^2) \end{cases}$$

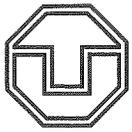
$$\int_0^{\infty} P_g''(\omega) \frac{d\omega}{\pi} + g = -\frac{g}{48\pi^2} (12z_F^2 - g^2) + \frac{z_F^3}{3\pi^2} = \frac{1}{3\pi^2} \left[ z_F^3 + \frac{g^3}{16} - \frac{3}{4} g z_F^2 \right]$$

$$E_x = \frac{1}{2} \text{Tr} \left( \int_0^{\infty} P_g \right) = \frac{1}{2} \int_0^{\infty} \frac{d^3 p}{(2\pi)^3} \left( \int_0^{\infty} \frac{d\omega}{\pi} P_g''(\omega) \right) = \frac{1}{2} \frac{4\pi}{8\pi^3} \int_0^{\infty} d\omega \frac{g^2}{g^2} \frac{1}{3\pi^2} \left[ z_F^3 + \frac{g^3}{16} - \frac{3}{4} g z_F^2 \right] =$$

$$= \frac{1}{3} \frac{1}{\pi} \frac{1}{2} \frac{1}{2\pi^2} \int_0^{\infty} d\omega \left( z_F^3 + \frac{g^3}{16} - \frac{3}{4} g z_F^2 \right) = \frac{2}{3} \frac{1}{\pi^3} \left( \frac{3}{4} \right) z_F^4 = \frac{1}{2} \frac{z_F^4}{\pi^3}$$

$$\xi_x = \frac{E_x}{\left( \frac{z_F^3}{3\pi^2} \right)} = \frac{1}{2} \frac{z_F^4}{\pi^3} \frac{(3\pi^2)^2}{z_F^3} = \frac{3}{2} \frac{z_F}{\pi} \checkmark$$

$$\int_0^{\infty} \frac{d^3 p}{(2\pi)^3} = \frac{4\pi}{8\pi^3} \int_0^{\infty} d\omega \frac{g^2}{g^2} = \frac{1}{2\pi^2} \int_0^{\infty} d\omega \frac{g^2}{g^2}$$



$$E_{xc}^{HF} = \frac{1}{2} \rho^2 \text{Tr}_\rho \left( \frac{N_s}{\rho} P_f \right) = \frac{1}{2} \rho^2 \frac{8\pi}{\rho^2} \text{Tr}_\rho (P_f) = \frac{1}{2} 8\pi \left( \sum_{z,s} M_{z,s} M_{z+s} - \sum_{z,s} M_{z,s} \right) = 4\pi \sum_{z,s} M_{z,s} M_{z+s} - 4\pi \rho$$

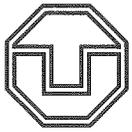
$$E_{xc}^{HF} = 4\pi \sum_{z,s} M_{z,s} M_{z+s} - \frac{4\pi}{\rho^2} \frac{3}{4\pi} = 4\pi \sum_{z,s} M_{z,s} M_{z+s} - \frac{3}{\rho^2}$$

Here we need:  $\text{Tr}_\rho (P_f) = \sum_{z,s} (M_{z,s} M_{z+s} - M_{z,s}) = - \sum_{z,s} (1 - M_{z,s}) M_{z+s}$

$$\text{Tr}_\rho (P_f) = \frac{1}{\rho} \sum_{i \in \mathbb{R}} P_f(i,s) = \int \frac{dx}{2\pi} M(x) P_f(x) = \int \frac{dx}{\pi} M(x) P_f''(x) = \int \frac{dx}{\pi} M(x) \left( N_s \int \frac{d^2 z}{(2\pi)^2} (-\pi) \delta(x + \frac{z}{2} - \frac{z+s}{2}) [f(\frac{z}{2}) - f(\frac{z+s}{2})] \right)$$

$$\text{Tr}_\rho (P_f) = -N_s \int \frac{d^2 z}{(2\pi)^2} M(\frac{z+s}{2} - \frac{z}{2}) [f(\frac{z}{2}) - f(\frac{z+s}{2})] = -N_s \int \frac{d^2 z}{(2\pi)^2} f(-\frac{z}{2}) f(\frac{z+s}{2}) = - \sum_{z,s} (1 - M_{z,s}) M_{z+s}$$

$$\boxed{\text{Tr}_\rho (P_f) = \sum_{z,s} M_{z,s} M_{z+s} - \rho}$$



Problem

$$P_f(z) \equiv \frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} \varphi_{\frac{z}{2}}(i\omega) \varphi_{\frac{z+\Omega}{2}}(i\omega+i\pi)$$

$$\begin{aligned} 1) \frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} P_f(i\Omega) &= \frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} \frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} \varphi_{\frac{i\Omega}{2}}(i\omega) \varphi_{\frac{i\Omega+\Omega}{2}}(i\omega+i\pi) = \frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} \varphi_{\frac{i\Omega}{2}}(i\omega) \frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} \varphi_{\frac{i\Omega+\Omega}{2}}(i\omega+i\pi) = \\ &= \underline{\underline{M_{\frac{\Omega}{2}} M_{\frac{\Omega}{2}+\Omega}}} \end{aligned}$$

$$2) \frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} P_f(i\Omega) = \int \frac{dz}{\sqrt{\pi}} M(z) P_f(z) = \int \frac{dx}{\sqrt{\pi}} M(x) P_f''(x)$$

$$P_f(z) = - \int \frac{dz}{\sqrt{\pi}} f(z) \varphi_{\frac{z}{2}}(z) \varphi_{\frac{z+\Omega}{2}}(z+i\Omega) = - \int \frac{dx}{\sqrt{\pi}} f(x) \left[ \varphi_{\frac{x}{2}}''(x) \varphi_{\frac{x+\Omega}{2}}(x+i\Omega) + \varphi_{\frac{x-\Omega}{2}}(x-i\Omega) \varphi_{\frac{x}{2}}''(x) \right]$$

$$P_f(\Omega+i\Omega) = - \int \frac{dx}{\sqrt{\pi}} \left[ f(x) \varphi_{\frac{x}{2}}''(x) \varphi_{\frac{x+\Omega}{2}}(x+\Omega) + f(x+\Omega) \varphi_{\frac{x}{2}}^*(x) \varphi_{\frac{x}{2}}''(x+\Omega) \right]$$

$$P_f''(\Omega+i\Omega) = - \int \frac{dx}{\sqrt{\pi}} [f(x) - f(x+\Omega)] \varphi_{\frac{x}{2}}''(x) \varphi_{\frac{x+\Omega}{2}}''(x+\Omega)$$

hence

$$\frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} P_f(i\Omega) = - \int \frac{dx}{\sqrt{\pi}} M(x) \int \frac{dy}{\sqrt{\pi}} [f(y) - f(y+x)] \varphi_{\frac{y}{2}}''(y) \varphi_{\frac{y+\Omega}{2}}''(y+x) = - \int \frac{dx}{\sqrt{\pi}} \int \frac{dy}{\sqrt{\pi}} M(x) [f(y) - f(y+x)] \times \varphi_{\frac{y}{2}}''(y) \varphi_{\frac{y+\Omega}{2}}''(y+x)$$

$$\begin{aligned} M(x) [f(y) - f(y+x)] &= \frac{1}{e^x - 1} \left[ \frac{1}{e^y + 1} - \frac{1}{e^{x+y} + 1} \right] = \frac{e^{x+y} - e^y}{(e^x - 1)(e^y + 1)(e^{x+y} + 1)} \\ &= \frac{e^y (e^x + 1)}{(e^x - 1)(e^y + 1)(e^{x+y} + 1)} = \frac{1}{(e^{-y} + 1)(e^{x+y} + 1)} = f(-y) f(x+y) \end{aligned}$$

$$\frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} P_f(i\Omega) = - \int \frac{dx}{\sqrt{\pi}} \int \frac{dy}{\sqrt{\pi}} f(-y) f(x+y) \varphi_{\frac{y}{2}}''(y) \varphi_{\frac{x+y}{2}}''(x+y) = - \underbrace{\int \frac{dy}{\sqrt{\pi}} (1 - f(y)) \varphi_{\frac{y}{2}}''(y)}_{-(1 - M_{\frac{\Omega}{2}})} \underbrace{\int \frac{dx}{\sqrt{\pi}} f(x+y) \varphi_{\frac{x+y}{2}}''(x+y)}_{M_{\frac{\Omega}{2}+\Omega}}$$

$$\frac{1}{\sqrt{\pi}} \sum_{i \in \mathbb{Z}} P_f(i\Omega) = - (1 - M_{\frac{\Omega}{2}}) M_{\frac{\Omega}{2}+\Omega} = \underline{\underline{- M_{\frac{\Omega}{2}+\Omega} + M_{\frac{\Omega}{2}} M_{\frac{\Omega}{2}+\Omega}}}$$